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Leggett–Williams type theorems with applications to nonlinear differential and integral equations

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A B S T R A C T

The aim of this article is to prove a few Leggett–Williams type theorems, in particular for a more general class of mappings than compact ones. We examine also invariant directions which satisfy a certain additional condition formulated in terms of a given seminorm. As applications of these results we prove a few existence results concerning positive solutions to Hammerstein integral equations or to a two-point boundary value problem.

1. Introduction

One of the classical results concerning the existence of positive invariant directions for compact mappings is the following theorem which originates from the work of Birkhoff and Kellogg [6].

Theorem 1. *Let U be a bounded open neighborhood of θ in an infinite-dimensional normed space E, and let $F: \partial U \to E$ be a compact mapping with $\|F(x)\| \geq \delta > 0$ for all $x \in \partial U$. Then F has an invariant direction, i.e. there is $x \in \partial U$ and $\lambda > 0$ such that $F(x) = \lambda x$.*

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The original result due to Birkhoff and Kellogg was established in a special case of the space of continuous real-valued functions defined on the unit interval, endowed with the L_2-norm, and the general case, according to our knowledge, was first stated by Schauder in a footnote in the paper [24] (the above formulation is cited after [13, Theorem 6.1, p. 62]).

An essential generalization of the Birkhoff–Kellogg criterion (in the version of Riedrich [21]) was established e.g. by Kryszewski (see [17, Theorem 3.1]). Since in this paper we are going to work mainly in ordered normed space, we recall another result established by Kryszewski in the same paper, which concerns invariant directions for positive mappings.

Theorem 2 (*[17, Theorem 4.1]*)**.** *Let E be an ordered Hausdorff topological vector space with a positive closed cone C_E being admissible. Suppose that D is a closed and star-shaped neighborhood of θ in E such that $\partial D \cap C_E \neq \emptyset$, and that $F: \partial D \cap C_E \to C_E$ is a positive compact operator for which there exists $y \in C_E \setminus A_D$, where $A_D = \bigcap_{t>0} tD$, such that*

$$\overline{F(\partial D \cap C_E)} \cap \{x \in E : x \in ty + (A_D \cap C_E),\ t \leq 0\} = \emptyset.$$

Then there exist $x_0 \in C_E \setminus \{\theta\}$ and $\lambda_0 > 0$ such that $F(x_0) = \lambda_0 x_0$.

It seems that from the point of view of possible applications, a little more convenient than Theorem 2 is the result due to Leggett and Williams [18] from 1977 (see Theorem 3 in Section 3).

All the above-mentioned results concern compact mappings satisfying some additional assumptions. The initial point of this paper is to prove some extensions of the Leggett–Williams theorem to a more general class of mappings than compact ones. We are going also to examine invariant directions satisfying some additional conditions formulated in terms of a given continuous seminorm. Let us recall that in the original Leggett–Williams theorem such an additional condition is stated in terms of a continuous, additive and positively homogeneous functional.

Another extension of the Leggett–Williams theorem to coupled invariant directions was proved by Borkar and Patil in [7]. In Section 4 we present a short proof of their result based on an application of the so-called reflection operator. It appears that in the coupled fixed point theory, reflection operators play an important role, that is, in that theory even quite sophisticated topological results can be quite easily proved with the help of such operators. In particular, in Section 4 we establish also a Vidossich type theorem for coupled fixed point sets for Volterra type operators by using reflection operators.

It appears that Leggett–Williams type theorems are useful tools to examine positive solutions to nonlinear differential and integral equations. In the second part of the paper we provide several applications of these theorems to results concerning the existence of positive solutions to Hammerstein integral equations (in finite dimensional spaces as well as in abstract Banach lattices), systems of Hammerstein equations and two-point boundary value problems.

2. Preliminaries

In this section we fix notation and recall some basic definitions and facts which will be used in the sequel.

By I throughout the paper we will denote a compact interval $[0, d] \subset \mathbb{R}$. Moreover, by μ_n we will denote the n-dimensional Lebesgue measure. If no confusion can arise, we will write simply μ.

We denote by $\overline{D}$ and ∂D the closure and boundary of the set D, respectively.

Let $(E, \|\cdot\|)$ be a normed space and let $|\cdot|$ be a continuous seminorm on E. By θ_E (or simply by θ) we will denote the zero element of E. By $B_E(x, r)$ and $B_{|\cdot|}(x, r)$ we will denote the closed ball in E centered at x and with radius $r > 0$, taken with respect to the norm $\|\cdot\|$ and the seminorm $|\cdot|$, respectively. Similarly, by $S_{|\cdot|}(x, r)$ we will denote the sphere $\{y \in E : |x - y| = r\}$.

By $L^p(\Omega)$, where $1 \leq p < +\infty$ and Ω is a Lebesgue measurable subset of $\mathbb{R}^n$, we will denote the Banach space of all complex-valued functions which are Lebesgue integrable with pth power. Moreover, if Ω is an open and bounded subset of $\mathbb{R}^n$, then by $C(\overline{\Omega}, \mathbb{R})$ we will denote the Banach space of all continuous real-valued functions defined on $\overline{\Omega}$, endowed with the supremum norm $\|\cdot\|_\infty$. Similarly, if K is a bounded and convex subset of a normed space and E is a Banach space, then by $BC(K, E)$ we will denote the Banach space of all bounded and continuous functions from K to E, endowed with the supremum norm. In particular, if $K = I$, then instead of $BC(K, E)$ we will write $C(I, E)$. Finally, the Banach algebra of continuous linear endomorphisms of E will be denoted by $\mathcal{L}(E)$.

The symbol '$\int$' denotes both the Lebesgue integral and the Bochner integral, if scalar- and vector-valued mappings are considered, respectively. (For basic properties of strongly measurable mappings and Bochner integrable mappings we refer the reader to [12].)

Now, we are going to recall some definitions concerning partially ordered structures.

Definition 1 (*Cf. [18, p. 249]*)**.** Let E be a real normed space. A non-empty closed and convex set $C_E \neq \{\theta\}$ contained in E is called a (positive) cone if the following conditions are satisfied:

(a) if $x \in C_E$, then $\lambda x \in C_E$ whenever $\lambda \geq 0$;
(b) if $x \in C_E$ and $-x \in C_E$, then $x = \theta$.

Notation. If C_E is a cone in a normed space E, then by $C_E(\theta, r)$ we will denote the intersection of C_E and $B_E(\theta, r)$, that is, $C_E(\theta, r) := C_E \cap B_E(\theta, r)$.