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On a nonlinear hyperbolic equation with a bistable reaction term

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ABSTRACT

Nonlinear second-order hyperbolic equations are gaining ground as models in many areas of application, as extensions of parabolic reaction–diffusion equations that might otherwise be used. The theory of travelling-wave solutions of such reaction–diffusion equations is well established. The present paper is concerned with its counterpart for the wider class of equations in the particular case that the reaction term is bistable. Conditions that are necessary and sufficient for the existence and uniqueness of these solutions are determined. A combination of traditional ordinary differential equation techniques and an innovatory integral equation approach is employed.

1. Nonlinear analysis

This paper is concerned with the analysis of wavefront solutions of the nonlinear equation

$$\varepsilon^2 u_{tt} + g(u)u_t = (k(u)u_x)_x + f(u), \tag{1.1}$$

in which subscripts denote partial differentiation, ε is a positive parameter that is not necessarily small, and f, g, and k are given functions. Equations of this form arise as mathematical models in many settings. These will be reviewed presently. When k and g are positive, the equation is of second-order hyperbolic type and can be classified as a *nonlinear telegraph equation*. In the limit $\varepsilon = 0$, this equation reduces to a *nonlinear reaction–diffusion equation* of second-order parabolic type. In that context, k is the diffusion coefficient and f represents the reaction term.

A solution of Eq. (1.1) of the form

$$u = \varphi(\eta) \quad \text{where } \eta = x - ct \tag{1.2}$$

and c is a constant, is called a *travelling-wave* solution. The constant c—which may be positive, negative, or zero—is known as the *wave-speed*. A *wavefront* solution is a travelling-wave solution that is defined for all $\eta \in \mathbb{R}$, monotonic, and such that

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$\varphi(\eta) \to u^{\pm}$ as $\eta \to \pm\infty$ for some numbers u^- and $u^+ \neq u^-$ that are equilibrium solutions of the equation. In other words, they are numbers for which $f(u^-) = f(u^+) = 0$. For convenience, it will be assumed that $u^- = 1$ and $u^+ = 0$. Thus

$$f(0) = f(1) = 0, \tag{1.3}$$

and the interest is in travelling-wave solutions for which

$$\varphi(\eta) \to 1 \quad \text{as } \eta \to -\infty, \tag{1.4}$$

and

$$\varphi(\eta) \to 0 \quad \text{as } \eta \to \infty. \tag{1.5}$$

In addition, since every wavefront solution of Eq. (1.1) satisfying (1.4) and (1.5) generates a one-parameter family of such solutions by translation with respect to the independent variable η, it will be assumed that

$$\varphi(0) = 1/2 \tag{1.6}$$

to avoid ambiguity in distinguishing genuinely different solutions.

Under the hypothesis that $f(u) > 0$ for $0 < u < 1$, wavefront solutions of Eq. (1.1) were recently the subject of investigation in [15]. In the terminology of the study of reaction–diffusion equations, this hypothesis means that f is such that of the two equilibrium solutions, $u = 0$ and $u = 1$, only the latter is stable; and, the reaction term is said to be monostable. The present paper deals with the case where

$$f(u) < 0 \quad \text{for } 0 < u < \alpha, \quad \text{and} \quad f(u) > 0 \quad \text{for } \alpha < u < 1, \tag{1.7}$$

for some $\alpha \in (0, 1)$. Under this hypothesis, both $u = 0$ and $u = 1$ are stable equilibria, and the reaction term is said to be bistable.

With the exception only of the case in which k is assumed to be constant, there appears to be little previous work on the existence of the wavefront solutions of equations of the class (1.1) when the reaction term is bistable. Moreover, the previous work seems to implicitly rely on the *a priori* assumption that the parameter ε and the wave-speed c are such that $c^2\varepsilon^2 < k$. Indicative is [17] in which besides k is constant, $c^2\varepsilon^2 < k$, (1.3) and (1.7), it is assumed that $f \in C^2([0, 1])$, $f'(0) < 0$, $f'(\alpha) > 0$, $f'(1) < 0$, $g \in C^1([0, 1])$, and $g(u) > 0$ for $0 \le u \le 1$. Such severe limitations will not be imposed in the present paper.

The precise assumptions to be employed are the following.

Hypothesis 1. The function $f \in C(0, \alpha) \cap C(\alpha, 1) \cap L^1(0, 1)$ satisfies (1.3) and (1.7), $g \in C(0, 1) \cap L^1(0, 1)$ is nonnegative and is positive somewhere in $(0, \alpha)$ and $(\alpha, 1)$, and $k \in C([0, 1])$ is nonnegative and not identically zero.

Thus, among other generalities on top of the obvious nonlinearity of the second-order term on the right-hand side of the equation, it is not assumed that f is continuous at 0, α, or 1, nor that f or g are anywhere differentiable.

The present work essentially establishes existence and non-existence results when $c^2\varepsilon^2 \ge k$ on $[0, 1]$, when $c^2\varepsilon^2 < k$ on $[0, 1]$, and, for all values of $c^2\varepsilon^2$ when k is monotonic. In each case, it will be established that the wavefront solution is unique. Furthermore, criteria characterizing how the conditions (1.4) and (1.5) are satisfied are determined.

The remainder of the paper is structured as follows. The next section summarizes the main results. The section thereafter comprises their proof. The final section discusses diverse applications of equations of the class (1.1).

2. Theory

Formally substituting (1.2) into (1.1) leads to the ordinary differential equation

$$c^2\varepsilon^2\varphi'' - cg(\varphi)\varphi' = (k(\varphi)\varphi')' + f(\varphi).$$

Defining

$$G(u) := \int_\alpha^u g(s)\,ds \quad \text{and} \quad K(u) := \int_\alpha^u \{k(s) - c^2\varepsilon^2\}\,ds,$$

this can be rewritten as

$$\big((K(\varphi))' + cG(\varphi)\big)' + f(\varphi) = 0. \tag{2.1}$$

By a travelling-wave solution of Eq. (1.1) with wave-speed c in an open interval I we shall understand a measurable function $\varphi : I \to [0, 1]$ such that $K(\varphi) \in C^1(I)$, $cG(\varphi) \in C(I)$, $(K(\varphi))' + cG(\varphi) \in W^{1,1}_{\text{loc}}(I)$, $f(\varphi) \in L^1_{\text{loc}}(I)$, and (2.1) holds almost everywhere in I. Any such function φ necessarily satisfies (2.1) classically in any open subinterval where $f(\varphi)$ is continuous. Moreover, φ is necessarily continuously differentiable in any open subinterval of I where $k(\varphi) \neq c^2\varepsilon^2$. On the other hand, in a subinterval where $k(\varphi) = c^2\varepsilon^2$, it is conceivable that φ is not differentiable, to the extent of not even being continuous. In this regard, (1.6) is to be interpreted as

$$\lim_{\eta \uparrow 0} \varphi(\eta) \ge 1/2 \ge \lim_{\eta \downarrow 0} \varphi(\eta). \tag{2.2}$$