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Energy decay for a nonlinear wave equation of variable coefficients with acoustic boundary conditions and a time-varying delay in the boundary feedback[☆]

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ABSTRACT

A variable-coefficient wave equation with acoustic boundary conditions and a time-varying delay in the boundary feedback is considered. Applying the Riemannian geometry method, we show that the decay rates of the nonlinear system with a time-varying delay are described by solutions to a first order ODE.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ (disjoint), where Γ_0 is nonempty and relatively open in Γ, $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. We consider the following initial boundary value problem:

$$\begin{cases} u_{tt} - \operatorname{div}(A(x)\nabla u) + \varphi(u_t) = 0 & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \Gamma_0 \times (0, \infty), \\ u_t + f(x)z_t + k(x)z = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu_A} - h(x)z_t + \mu_1\beta(u_t(x,t)) + \mu_2 u_t(x, t-\tau(t)) = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ z(x,0) = z_0(x), & \text{on } \Gamma_1, \\ u_t(x, t-\tau(0)) = j_0(x, t-\tau(0)), & \text{on } \Gamma_1 \times (0, \tau(0)), \end{cases} \tag{1.1}$$

where $\operatorname{div} X$ denotes the divergence of the vector field X in the Euclidean metric, $A(x) = (a_{ij}(x))$ are symmetric and positive definite matrices for all $x \in \mathbb{R}^n$ and $a_{ij}(x)$ are smooth functions on $\mathbb{R}^n$. $\frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \nu_i$, where $\nu = (\nu_1, \nu_2, \ldots, \nu_n)^T$ denotes the outward unit normal vector of the boundary and $\nu_A = A\nu$. $f, k, h : \overline{\Gamma}_1 \to \mathbb{R}$ and $\varphi, \beta : \mathbb{R} \to \mathbb{R}$ are given functions.

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Here, $\tau(t) > 0$ is a time-varying delay, μ_1, μ_2 are real numbers with $\mu_1 > 0, \mu_2 \neq 0$ and the initial data (u_0, u_1, j_0, z_0) belongs to a suitable space.

On the function $\tau(t)$, we assume that there exist positive constants τ_0, τ_1 such that

$$0 < \tau_1 \le \tau(t) \le \tau_0, \quad \forall t > 0. \tag{1.2}$$

Moreover, we assume

$$d_1 \le \tau'(t) \le d < 1, \quad \forall t > 0, \tag{1.3}$$

where d, d_1 are constants and

$$\tau \in W^{2,\infty}([0, T]), \quad \forall T > 0. \tag{1.4}$$

If $t < \tau(t)$, then $u_t(t - \tau(t))$ is in the past and we need an initial value in the past. Moreover, using the mean value theorem, by (1.3) we have

$$\tau(t) - \tau(0) < t.$$

That is

$$t - \tau(t) > -\tau(0).$$

We thus obtain the initial condition

$$u_t(x, t - \tau(0)) = j_0(x, t - \tau(0)), \quad \text{on } \Gamma_1 \times (0, \tau(0)).$$

The system (1.1) represents a wave equation with acoustic boundary conditions which is related to noise control and suppression in practical applications. The noise sound propagates through some acoustic medium, for example, through air in a room which is characterized by a bounded domain Ω and whose walls, ceiling and floor are described by the boundary conditions. This is a coupled system of second and first order in time partial differential equations, where the coupling is given on the portion Γ_1 of the boundary. The PDE system with acoustic boundary conditions was introduced by Morse and Ingard [18] and developed by Beale and Rosencrans [1–3]. The problem of well-posedness and stabilization of the wave equation has been widely investigated, see e.g. [15,14,10,4,6,5]. In particular, many such results concerning the wave system with acoustic boundary conditions are also available in the literature. See, for instance [13,27,11,12,30] and references therein.

On the other hand, time delays arise in many applications because most phenomena naturally depend not only on the present state but also on the past history of the system in a more complicated way. In recent years, different equations with time delay effects have become an active area of applied mathematics due to physical reasons with non-instant transmission phenomena as high velocity fields in wind tunnel experiments, or other memory processes, and specially biological motivations like species' growth or incubating time on disease models among many others. For the constant delay, see [9,19,20,26,24] and references therein. For the time-varying delay, see [23,22,21,25]. Introduction of the delay in the model enriches its dynamics and allows a precise description of the real life phenomena. Therefore, in this paper, a time-varying delay is introduced into the boundary feedback of system (1.1). It is well-known that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay, see [9]. The stability issue of a structurally acoustic model with a delay term is, therefore, of theoretical and practical importance. We are interested in the effect of a time-varying delay in boundary stabilization of the structurally acoustic model (1.1).

When $A(x) \equiv I$, we say that the system (1.1) is of constant coefficients. In this case, many energy decay results on such problems are available in the literature, see [13,27,11,12,9,19,20,23,22,21]. Generally, the coefficients matrix $A(x)$ is related to the material in applications. In this paper, we consider the system (1.1) of variable coefficients with a general $A(x)$. The wave equations with variable coefficients arise in mathematical modeling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetic, fluid flows through porous media (e.g. modeling traveling waves in a inhomogeneous gas), and other areas of physics and engineering. For the variable-coefficient system (1.1), the main tool we use is the Riemannian geometry method, which was introduced by P.F. Yao [34] to deal with controllability for the variable-coefficient wave equation and has been widely used in the literature, for instance, see [30,26,24,25,16,29,7,8,31,32,35,17]. This is a new and powerful method to deal with variable-coefficient problems. For a survey on the Riemannian geometry method, see [34,35].

Whether in constant-coefficient case or in variable-coefficient case, the asymptotic behavior of the wave equation with a delay term is more considered in the linear system where exponential stability result can be proved. The nonlinear system, especially the essentially nonlinear system, is scarce investigated. In this paper, the decay rates for the solution of nonlinear system (1.1) with a time-varying delay are described by solutions of nonlinear, dissipative ODE given by (2.11) (as in [14]). In several situations, the obtained ODE can be easily solved (see [5, Section 8]) and the decay rates can be given explicitly. As far as we know, the decay result is hardly seen in current literature on the study of constant/variable-coefficient nonlinear wave systems with delay effects.

In the case of variable coefficients, the wave systems with either acoustic boundary conditions or a delay term in the feedback have been investigated by [30,26,24,25,17]. However, in the present work, we will consider a nonlinear wave system simultaneously with acoustic boundary conditions and a time-varying delay term in the boundary feedback. The