Nonlinear Analysis 112 (2015) 156–164

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Gradient estimates for doubly nonlinear diffusion equations

Daguang Chen, Changwei Xiong *

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, PR China

A R T I C L E I N F O

Article history:
Received 6 June 2014
Accepted 26 August 2014
Communicated by Enzo Mitidieri

Keywords:
Doubly nonlinear diffusion equation
Gradient estimate
Harnack inequalities

A B S T R A C T

In this short note we obtain some gradient estimates for the positive solution to the doubly nonlinear diffusion equation on closed Riemannian manifold with Ricci curvature bounded below by a non-positive constant. As applications, we also derive corresponding Harnack inequalities.

1. Introduction

In this paper, we study the following doubly nonlinear diffusion equation (DNDE for short)

$$u_t = \Delta_p(u^\gamma) = div(|\nabla(u^\gamma)|^{p-2}\nabla(u^\gamma)), \tag{1.1}$$

where $\gamma > 0, p > 1$ and Δ_p is the p-Laplacian. This equation appears in several models, for example in non-Newtonian fluids, glaciology and turbulent flows in porous media [15]. And it has some special cases, which have been receiving many attentions. Other than the classical heat equation ($p = 2, \gamma = 1$), the porous medium equation ($p = 2, \gamma > 1$), fast diffusion equation ($p = 2, \gamma < 1$) and p-Laplacian heat equation ($\gamma = 1$) are important examples.

Let (M^n, g) be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by $-K$, where $K \geq 0$. For the positive solution of the heat equation

$$u_t = \Delta u, \tag{1.2}$$

Li and Yau [11] obtained the celebrated gradient estimate:

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq \frac{n\alpha^2 K}{2(\alpha-1)} + \frac{n\alpha^2}{2t}, \tag{1.3}$$

where $\alpha > 1$ is a constant.

Since then, there have arisen various gradient estimates for the heat equation (1.2). In [4], Davies improved Li–Yau's estimate (1.3) to

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq \frac{n\alpha^2 K}{4(\alpha-1)} + \frac{n\alpha^2}{2t}.$$

* Corresponding author. Tel.: +86 01051534046.
E-mail addresses: dgchen@math.tsinghua.edu.cn (D. Chen), xiongcw10@mails.tsinghua.edu.cn (C. Xiong).

http://dx.doi.org/10.1016/j.na.2014.08.017
0362-546X/

And more recently, Li and Xu [10] got a new gradient estimate

$$\frac{|\nabla u|^2}{u^2} - \left(1 + \frac{\sinh(Kt)\cosh(Kt) - Kt}{\sinh^2(Kt)}\right)\frac{u_t}{u} \le \frac{nK}{2}(\coth(Kt) + 1),$$

and its linearized version (see also [2,14])

$$\frac{|\nabla u|^2}{u^2} - \left(1 + \frac{2}{3}Kt\right)\frac{u_t}{u} \le \frac{n}{2}\left(\frac{1}{t} + K + \frac{1}{3}K^2 t\right).$$

In [7], Hamilton derived another type estimate for the positive solution of (1.2)

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt}\frac{u_t}{u} \le e^{4Kt}\frac{n}{2t}.$$

For the gradient estimates for the positive solution of the porous medium equation ($p = 2, \gamma > 1$) and the p-Laplace equation ($\gamma = 1$) and relevant applications, one can refer to [1,3,8,9,12,18–22] and the references therein.

In this short note, by using the gradient estimate, we obtain several type estimates for DNDE (1.1).

Theorem 1.1. *Let M^n be a closed Riemannian manifold with $Ric \ge -K$ for some constant $K \ge 0$. Suppose $u(t)$ is a positive solution of the DNDE heat equation*

$$u_t = \Delta_p(u^\gamma) = div(|\nabla(u^\gamma)|^{p-2}\nabla(u^\gamma)), \quad \text{on } M \times (0, T],$$

with $b = \gamma - \frac{1}{p-1} > 0$. Set $v = \frac{\gamma}{b}u^b$, $L = \sup_{M\times(0,T]} \frac{p}{2}bv|\nabla v|^{p-2}$ and $a = \frac{p}{bn}$. Then we have

$$\frac{|\nabla v|^p}{v} - \alpha(t)\frac{v_t}{v} \le \varphi(t), \tag{1.4}$$

where $\alpha(t)$ and $\varphi(t)$ are the following:

1. *Li–Xu type:*

$$\begin{aligned} \alpha(t) &= 1 + \frac{\sinh(KLt)\cosh(KLt) - KLt}{\sinh^2(KLt)}, \\ \varphi(t) &= \frac{KL}{a}(\coth(KLt) + 1); \end{aligned} \tag{1.5}$$

2. *Linearized Li–Xu type:*

$$\alpha(t) = 1 + \frac{2}{3}KLt, \qquad \varphi(t) = \frac{1}{a}\left(\frac{1}{t} + KL + \frac{1}{3}(KL)^2 t\right); \tag{1.6}$$

3. *Hamilton type:*

$$\alpha(t) = e^{2KLt}, \qquad \varphi(t) = \frac{1}{at}e^{4KLt}, \quad \text{for } 0 < KLt < 1; \tag{1.7}$$

4. *Davies type:*

$$\alpha(t) = constant > 1, \qquad \varphi(t) = \frac{1}{a}\left(\frac{\alpha^2 KL}{2(\alpha - 1)} + \frac{\alpha^2}{t}\right). \tag{1.8}$$

Remark 1.2. Inspecting the following expansions of $\alpha(t)$ and $\varphi(t)$ in Li–Xu type estimate, one can compare (1.5) and (1.6)

$$\frac{KL}{a}(\coth(KLt) + 1) = \frac{1}{a}\left(\frac{1}{t} + KL + \frac{1}{3}(KL)^2 t - \frac{1}{45}(KL)^4 t^3 + O(t^5)\right),$$

$$1 + \frac{\sinh(KLt)\cosh(KLt) - KLt}{\sinh^2(KLt)} = 1 + \frac{2}{3}KLt - \frac{4}{45}(KL)^3 t^3 + O(t^5).$$

Therefore, (1.6) are the leading terms of the expansion of (1.5), which implies that Li–Xu type estimate is sharper than its linearized version.

Remark 1.3. When $\gamma \to 1$ and $p \to 2$, these estimates reduce to the corresponding ones for the classical heat equation. Take Hamilton type estimate for example. As $\gamma \to 1$ and $p \to 2$, $b \to 0$. Therefore it follows that

$$bv = \gamma u^b \to 1,$$