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Second derivative estimates for uniformly elliptic operators on Riemannian manifolds

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ABSTRACT

In this paper, we obtain a uniform $W^{2,\varepsilon}$-estimate of solutions to the fully nonlinear uniformly elliptic equations on Riemannian manifolds with a lower bound of sectional curvature using the ABP method.

1. Introduction

We study regularity estimates for solutions to a class of the fully nonlinear uniformly elliptic equations

$$F(D^2u, x) = f \quad \text{in } B_R(z_0) \subset M \tag{1}$$

on a complete Riemannian manifold M, where the operator F satisfies the hypothesis (H1). Under the assumption that sectional curvature of the underlying manifold M is nonnegative, the Krylov–Safonov Harnack estimate [13] was initiated by Cabré in his paper [2], where a priori global Harnack inequality for linear elliptic equations was established by obtaining the Aleksandrov–Bakelman–Pucci (ABP) estimate on M. Later, Kim [10] improved Cabré's result removing the sectional curvature assumption and imposing the certain conditions on the squared distance function. Recently, Wang and Zhang [22] proved a version of the ABP estimate on M with a lower bound of Ricci curvature, and hence a locally uniform Harnack inequality for nonlinear elliptic operators on M provided that the sectional curvature is bounded from below. A priori Harnack estimate has been extended in [12] for viscosity solutions using the regularization of Jensen's sup-convolution on Riemannian manifolds. The Hölder continuity is obtained as an immediate consequence of the Harnack inequality. In [11,12], the parabolic Harnack inequality and the ABP–Krylov–Tso type estimate were established on the Riemannian manifolds with a lower curvature bound.

This paper is concerned with a uniform estimate for second order derivatives of solutions to (1) on Riemannian manifolds with a nonpositive lower bound of sectional curvature. In the Euclidean space, a uniform $W^{2,\varepsilon}$-estimate (for some $\varepsilon > 0$) for linear, nondivergent elliptic operators with measurable coefficients was first discovered by Lin [15]. It is known that for any $p \geq 1$, a uniform $W^{2,p}$-estimate for uniformly elliptic equations with measurable coefficients is not valid; see [16,19]. In [3] and [4, Chapter 7], Caffarelli dealt with the $W^{2,\varepsilon}$-estimate for fully nonlinear elliptic operators with measurable coefficients whose proof relies on the ABP estimate, where the ABP estimate proved by Aleksandrov, Bakelman, and Pucci in sixties has played a crucial role in the Krylov–Safonov theory and in the development of the regularity theory for fully nonlinear elliptic equations. On the other hand, $W^{2,p}$-estimates ($n < p < +\infty$) of Calderón and Zygmund for linear elliptic operators with

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continuous coefficients were extended in [4] for convex fully nonlinear elliptic equations which are continuous in spatial variables as a perturbation theory by accelerating a decay rate of the measure estimates for the Hessian of the solutions. Based on the Calderón–Zygmund estimates, potential theoretic estimates for the fundamental solution to nondivergent elliptic equations were established in [8], which turned out to be closely tied to the Harnack estimate.

In this paper, we investigate a uniform $W^{2,\varepsilon}$-regularity (for some $\varepsilon > 0$) for fully nonlinear elliptic operators on Riemannian manifolds. Making use of the ABP type estimate on Riemannian manifolds, we follow Caffarelli's approach under the assumption that sectional curvature is bounded from below. It can be checked that a straightforward adaptation of the Euclidean method yields the $W^{2,\varepsilon}$-estimate on Hadamard manifolds which are complete and simply connected Riemannian manifolds with nonpositive sectional curvature everywhere. In general, it is not applicable directly due to the existence of the cut locus. Indeed, it is difficult to use the squared distance functions as global test functions as in the Euclidean case. To proceed with the ABP method, we introduce the notion of the special contact set in Definition 3.14. The special contact set consists of the points where the solution has a global tangent function from below, which is a sum of the barrier functions constructed in Lemma 3.5 and squared distance functions. With the help of a standard scaling argument via the Calderón–Zygmund technique, the notion of the special contact set enables us to employ an iterative procedure based on the scale-invariant ABP type estimate in Proposition 3.15. Therefore we obtain a (locally) uniform $W^{2,\varepsilon}$-estimate for a class $\mathcal{S}^*$ of solutions to fully nonlinear uniformly elliptic equations; see Definition 2.8. Along the lines of the Euclidean Calderón–Zygmund theory, we are interested in a potential theory for nondivergent elliptic operators on Riemannian manifolds, and its relation to the Harnack inequality with a certain curvature bound of the underlying manifolds, which we hope to consider in subsequent works. Lastly, we end the introduction by stating our result as follows.

Theorem 1.1 ($W^{2,\varepsilon}$-*Estimate*)**.** *Let M be a complete Riemannian manifold with the sectional curvature bounded from below by $-\kappa$ for $\kappa \geq 0$. Let $0 < R \leq R_0$ and $x_0 \in M$ and $f \in L^{n\eta}(B_{2R}(x_0))$ for $\eta := 1 + \log_2 \cosh(4\sqrt{\kappa}R_0)$. There exist uniform constant $\varepsilon > 0$ and $C > 0$ such that if a smooth function u belongs to $\mathcal{S}^*(\lambda, \Lambda, f)$ in $B_{2R}(x_0)$, then we have that $u \in W^{2,\varepsilon}(B_R(x_0))$ with the estimate*

$$\left(\fint_{B_R(x_0)} |u|^{\varepsilon} + |R\nabla u|^{\varepsilon} + \left|R^2 D^2 u\right|^{\varepsilon}\right)^{\frac{1}{\varepsilon}} \leq C\left\{\|u\|_{L^{\infty}(B_{2R}(x_0))} + \left(\fint_{B_{2R}(x_0)} |R^2 f|^{n\eta}\right)^{\frac{1}{n\eta}}\right\},$$

where $\varepsilon > 0$ and $C > 0$ depend only on n, λ, Λ, and $\sqrt{\kappa}R_0$, and we denote $\fint_Q f := \frac{1}{\mathrm{Vol}(Q)}\int_Q f \, d\,\mathrm{Vol}$.

Here, the $W^{2,\epsilon}$-estimate is scale-invariant in the sense that the constants $\epsilon, C > 0$ and $\eta \geq 1$ depend only on dimension, the ellipticity constants and a scale-invariant curvature bound from below. We remark that $L^{n\eta}$ is a natural Lebesgue space for uniformly elliptic equations on Riemannian manifolds with a lower Ricci curvature bound due to Bishop–Gromov's Theorem 2.1; see Lemma 2.2. In particular, for the case when a Riemannian manifold has nonnegative sectional curvature, i.e., $\kappa = 0$, the $W^{2,\varepsilon}$-estimate is global, and depends only on dimension n, and the ellipticity constants λ, and Λ with $\eta = 1$.

2. Preliminaries

Throughout this paper, let (M, g) be a smooth, complete Riemannian manifold of dimension n, where g is the Riemannian metric. A Riemannian metric defines a scalar product and a norm on each tangent space, i.e., $\langle X, Y\rangle_x := g_x(X, Y)$ and $|X|_x^2 := \langle X, X\rangle_x$ for $X, Y \in T_xM$, where T_xM is the tangent space at $x \in M$. Let $d(\cdot,\cdot)$ be the Riemannian distance on M. For a given point $y \in M$, $d_y(x)$ denotes the distance to x from y, i.e., $d_y(x) := d(x, y)$. A Riemannian manifold is equipped with the Riemannian measure $\mathrm{Vol} = \mathrm{Vol}_g$ on M which is denoted by $|\cdot|$ for simplicity.

For a smooth function $u : M \to \mathbb{R}$, the gradient ∇u of u is defined by

$$\langle \nabla u, X\rangle := du(X)$$

for any vector field X on M, where $du : TM \to \mathbb{R}$ is the differential of u. The Hessian D^2u of u is defined as

$$D^2u\,(X, Y) := \langle \nabla_X \nabla u, Y\rangle\,,$$

for any vector fields X, Y on M, where ∇ denotes the Riemannian connection of M. We observe that the Hessian D^2u is a symmetric 2-tensor over M, and $D^2u(X, Y)$ at $x \in M$ depends only on the values X, Y at x, and u in a small neighborhood of x. By the metric, the Hessian of u at x is canonically identified with a symmetric endomorphism of T_xM:

$$D^2u(x)\cdot X = \nabla_X \nabla u, \quad \forall X \in T_xM.$$

We will write $D^2u(x)\,(X, Y) = \left\langle D^2u(x)\cdot X,\ Y\right\rangle$ for $X \in T_xM$. In terms of local coordinates $\left(x^i\right)$ of M, the components of D^2u are written by

$$\left(D^2u\right)_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k}$$