Nonlinear Analysis 106 (2014) 18–34

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Bending and stretching energies in a rectangular plate modeling suspension bridges

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ARTICLE INFO

Article history:
Received 25 January 2014
Accepted 16 April 2014
Communicated by Enzo Mitidieri

MSC:
35G30
74B20
35J35

Keywords:
Nonlinear elasticity
Prestressed plates
Bending and stretching energies
Variational methods

ABSTRACT

A rectangular plate modeling the roadway of a suspension bridge is considered. Both the contributions of the bending and stretching energies are analyzed. The latter plays an important role due to the presence of the free edges. A linear model is first considered; in this case, separation of variables is used to determine explicitly the deformation of the plate in terms of the vertical load. Moreover, the same method allows us to study the spectrum of the linear operator and the least eigenvalue. Then the stretching energy is introduced without linearization and the equation becomes quasilinear; the nonlinear term also affects the boundary conditions. We consider two quasilinear models; the *surface increment model* (SIM) in which the stretching energy is proportional to the increment of the surface and a *nonlocal model* (NLM) introduced by Berger in the 50s (see Berger (1955)). The SIM and the NLM are studied in detail. According to the strength of prestressing we prove the existence of multiple equilibrium positions.

1. Introduction

Consider a rectangular plate hinged at two opposite edges and free on the remaining two edges. We have in mind a suspension bridge and our purpose is to study several different models to describe its behavior. We view the roadway of the bridge as a long narrow rectangular thin plate, hinged on its short edges where the bridge is supported by the ground, and free on its long edges. Let L denote its length and 2ℓ denote its width; a realistic assumption is that $2\ell \cong \frac{L}{100}$. For simplicity, we take $L = \pi$ so that, in the sequel,

$$\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2. \tag{1}$$

This model is considered in [1] where the analysis of the bending energy of the plate leads to a fourth order elliptic equation. However, motivated by the presence of free parts of the boundary, the stretching energy was neglected. A more accurate analysis would have to take account of the stretching energy. From a mathematical point of view, one may notice that $H^2 \subset H^1$ and that the H^2-norm bounds the H^1-norm; whence, the stretching energy may be considered as a "lower order term" when compared with the bending energy. But, as we shall see, the former plays an important role in the model and a deep motivation to introduce the stretching energy comes from structural engineering and physics. Concrete is weakly elastic and heavy loads can produce cracks. On the other hand, metals are more elastic and react to loads by bending.

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http://dx.doi.org/10.1016/j.na.2014.04.011
0362-546X/

For this reason, prestressed concrete structures have been conceived. According to [2, p. 28], the father of prestressed concrete bridges is the French engineer Eugène Freyssinet (1879–1962) and these bridges were first built some 75 years ago. Prestressing metal tendons (generally of high tensile steel) are used to provide a clamping load which produces a compressive stress that balances the tensile stress that the concrete compression member would otherwise experience due to a load. Prestressed concrete is obtained by casting concrete around tensioned tendons; this method produces a winning interaction between tendons and concrete. It protects the tendons from corrosion and allows for direct transfer of tension. The concrete adheres and bonds to the bars, and when the tension is released it is transferred to the concrete as compression by static friction. This technique creates an "elasticized concrete" and explains on the one hand why stretching is negligible when computing the total energy of the plate Ω, and on the other hand why it should not be neglected if the purpose is to analyze a more precise model.

In the present paper we study the plate model by considering also the stretching energy. First we consider a linear model and we prove that the problem is well-posed. With a separation of variables method we also determine the explicit solution and this allows to quantify the cross behavior of the plate, a phenomenon clearly visible in the Tacoma Narrows Bridge collapse [3]. Next, we analyze the competing effect between bending and stretching, which occurs in prestressed structures; this brings us to study an eigenvalue problem and its spectrum. With these results, the linear theory seems to be sufficiently clear. However, in recent years, the nonlinear structural behavior of suspension bridges has been uncovered, see e.g. [4–7]. Therefore, a linear model may not be sufficiently accurate to describe the static behavior of bridges, especially for large deflections as the ones visible in the video [3]. Although mathematical models in nonlinear elasticity are fairly complicated it seems that their use is unavoidable. In an important paper, dated some decades ago, Gurtin [8] showed the necessity of nonlinear models in elasticity and concludes his work by writing

Our discussion demonstrates why this theory is far more difficult than most nonlinear theories of mathematical physics. It is hoped that these notes will convince analysts that nonlinear elasticity is a fertile field in which to work.

Since fully nonlinear plate equations appear intractable, and since linear equations fail to highlight important phenomena, a first step should be to study models having some nonlinearity only in the lower order terms. This appears to be a good compromise between too poor linear models and too complicated fully nonlinear models. This compromise is quite common in elasticity, see e.g. the book by Ciarlet [9, p. 322] who describes the method of asymptotic expansions for the thickness ε of a plate as a "partial linearization"

in that a system of quasilinear partial differential equations, i.e., with nonlinearities in the higher order terms, is replaced as $\varepsilon \to 0$ *by a system of semilinear partial differential equations, i.e., with nonlinearities only in the lower order terms.*

In this paper we make one further step towards fully nonlinear models. Instead of a semilinear model, where the nonlinearities merely appear in the zero order term of the differential equation, we consider two quasilinear models with nonlinearities involving derivatives of the unknown function (the vertical displacement of the plate). The Euler–Lagrange equation contains second order nonlinear differential operators while the highest order operator (fourth order) remains linear. We first consider the stretching energy as a multiple of the increment of the surface and not just its first order asymptotic approximation which is usually employed to describe small displacements of the plate. Then we consider a nonlocal quasilinear model NLM going back to Berger [10] which may be seen as a second order approximation of the stretching energy. For both these models we derive a quasilinear Euler–Lagrange equation and we prove existence and multiplicity results depending on the strength of prestressing.

This paper is organized as follows. In Section 2 we recall the classical model for the energy of an elastic plate. In Section 3 we study the linearized model and we state the corresponding results: well-posedness (Theorem 1), the explicit form of the solution and behavior of the cross bending (Theorem 2), analysis of the least eigenvalue and of the whole spectrum of the linearized bending–stretching competition (Theorem 4). In Section 4 we consider the quasilinear SIM: after deriving the Euler–Lagrange equation (30) from the minimization of the energy, we prove existence and multiplicity results for both the homogeneous problem (Theorem 5) and the inhomogeneous problem (Theorem 6). In Section 5 we introduce the NLM by adapting the Berger model to our partially hinged plate: we derive the Euler–Lagrange equation (38) from the minimization of the energy, then we prove existence and multiplicity results for both the homogeneous problem (Theorem 7) and the inhomogeneous problem (Theorem 8). Finally, Sections 6–11 are devoted to the proofs of the results.

2. A model for a partially hinged plate

Stretching occurs when the horizontal position of the plate is fixed on the boundary $\partial\Omega$ and a deformation of Ω yields a variation of its surface. In our case, the plate is fixed on the two short edges and a deformation of the plate necessarily yields a variation of its surface. von Kármán [11] assumes that the elastic force is proportional to the increment of the surface in such a way that the stretching energy for the plate, whose vertical deflection is u, reads

$$\mathbb{E}_S(u) = \int_\Omega \left(\sqrt{1+|\nabla u|^2} - 1\right) dxdy. \tag{2}$$

For small deformations u, the asymptotic expansion $(\sqrt{1+|\nabla u|^2} - 1) \sim |\nabla u|^2/2$ leads to the Dirichlet integral

$$\mathbb{E}_S(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dxdy. \tag{3}$$