Nonlinear Analysis 106 (2014) 35–56

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Green's function of a chemotaxis model in the half space and its applications

Renkun Shi *

Department of Mathematics, Shanghai Jiao Tong University, Shanghai, 200240, PR China

ARTICLE INFO

Article history:
Received 24 January 2014
Accepted 15 April 2014
Communicated by Enzo Mitidieri

Keywords:
Chemotaxis
Half space
Green's function
Decay rate

ABSTRACT

In this paper, we are interested in a degenerate Keller–Segel model in the half space $x_n > lt$ in high dimensions, with a constant l. Under the conservative boundary condition, we study the global solvability, regularity and long time behavior of solutions. We first construct Green's function of the half space problem and study its properties. Then by Green's function, we give the implicit formula of solutions. We prove that when the initial data is small enough, the half space problem is always globally and classically solvable. We also obtain decay estimates of solutions when the parameter l have different signs. For $l < 0$, the time decay of solutions is $(1+t)^{-n/2}$. While for $l > 0$, the time decay is $(1+t)^{-(n-1)/2}$, and in this case, we also obtain an exponential decay in the spatial space in the x_n-direction.

1. Introduction

Chemotaxis is a phenomenon describing the movements of cells, bacteria or organisms in response to some chemical signals. A classical chemotaxis model is the well-known Keller–Segel (KS) model [1] proposed by Keller and Segel in 1970s. It has a very important simplified version described as:

$$\begin{cases} \partial_t u - \Delta u = -\beta \nabla \cdot (u \nabla v), \\ \lambda v - \Delta v = u, \end{cases} \tag{1.1}$$

where $x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n$ are space variables, $u(x,t)$ denotes the density of cells, and $v(x,t)$ denotes the chemical concentration. The parameters β represents the sensitivity of cells to the chemical signals, and $\beta > 0\,(<0)$ corresponds to attractive (repulsive) chemotaxis. λ is a positive parameter.

This chemotaxis model has received lots of attention due to its various meaningful issues and significance in biology. One interesting feature of the model lies in the competition between diffusion and nonlocal aggregation, and consequently, the solutions of the model can behave very differently. So far, there have been much literature that investigated it from various viewpoints. When $n = 1$, the solutions of the model exist globally and are uniformly bounded (see [2]). When $n \geq 2$, there can be more interesting phenomenon that the finite time or infinite time blow-up may occur (see in [3–7]). For the problem (1.1) with small initial data, global existence of solutions always hold and the long time behavior is like the heat equation, for which, one can refer to [8–10].

The papers mentioned above are devoted to Cauchy problems and initial–boundary value problems in a bounded domain. While for a moving boundary problem, to the author's best knowledge, such relative results are not shown so far. In this

* Tel.: +86 13818081609.
E-mail addresses: srk0221@aliyun.com, blalbible@163.com.

http://dx.doi.org/10.1016/j.na.2014.04.009
0362-546X/© 2014 Elsevier Ltd. All rights reserved.

paper, we are interested in a half space problem of the Keller–Segel model, and we study the regularity and long time behavior of its solutions. We consider the problem in the half space $x_n > lt$ in high dimensions with some given constant l, and correspondingly, $x_n = lt$ is the moving boundary.

$$\begin{cases} \partial_t u - \Delta u = -\beta \nabla \cdot (u \nabla v), & x \in \mathbb{R}^n, x_n > lt, t > 0, \\ \lambda v - \Delta v = u, & x \in \mathbb{R}^n, x_n > lt, t > 0, \\ (l + \partial_{x_n}) u \,|_{x_n = lt} = \partial_{x_n} v \,|_{x_n = lt} = 0, & \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n, x_n > 0. \end{cases} \tag{1.2}$$

For the convenience, we apply the coordinate transform $x_n \to x_n + lt$ in (1.2), and obtain

$$\begin{cases} \partial_t u - l \partial_{x_n} u - \Delta u = -\beta \nabla \cdot (u \nabla v), & x \in \mathbb{R}^n_+, t > 0, \\ \lambda v - \Delta v = u, & x \in \mathbb{R}^n_+, t > 0, \\ (l + \partial_{x_n}) u \,|_{x_n = 0} = \partial_{x_n} v \,|_{x_n = 0} = 0, & \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n_+, \end{cases} \tag{1.3}$$

where $\mathbb{R}^n_+ = \{(x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n | x_n > 0\}$ and $n \geq 2$. Here we pose the conservative boundary condition for $u(x, t)$ to preserve the total mass of cells, which is a significant feature of chemotaxis models.

Usually, when we consider an initial-boundary value problem, the energy estimate is needed. However, pointwise estimates cannot be obtained from purely energy method. This technical difficulty can be partly dealt with well by the Fourier analysis in the whole space. In that case, we often use Green's functions of equations to formulate solutions, which is very helpful for our studies. And the key point is applying the Fourier analysis to Green's functions to obtain some useful estimates, such technique can be seen in many works [11–14]. While for the half space problem, Green's functions are not so easy to yield, since the traditional Fourier analysis is valid only in the whole space. In this paper, we first aim at constructing Green's function of (1.3) and investigating its properties. The process mainly follows Liu and Yu [15] and Wang and Yu [16] which applied the standard Fourier–Laplace transforms to study the boundary relations. Their idea is to convert the differential equations into algebraic forms and then to use complex analysis and the boundary relation to invert the transforms. The procedure is called algebraic-complex scheme [16]. In constructing Green's function of (1.3), we also find a useful identity, by which Green's function can be written in a beautiful and symmetrical form. The identity is also very helpful for our estimates. We shall show that in Section 2.

The second aim of this paper is to study the global existence, regularity of solutions and long time behavior of the system (1.3) for the parameter l with different signs. We prove that no matter what sign of l, the initial–boundary value problem (1.3) is always globally and classically solvable for small initial data. We mention that the initial data $u_0(x)$ considered here is mathematically general and not restricted to be positive, although the density $u(x, t)$ is always thought nonnegative. This consideration is also meaningful when we proceed further studies about perturbation stability or some other problems. More than that, we obtain the decay estimates of solutions in L^∞ norm for different l. When l is negative, the time decay rate is $(1 + t)^{-n/2}$, and the coefficient is uniform about l, which is a little beyond expectation. While for positive l, there is a more interesting result that the solutions can be divided into two parts having different behaviors. For one part, the time decay rate remains $(1+t)^{-n/2}$, while for the other one, the time decay changes to $(1+t)^{-(n-1)/2}$ and the spatial behavior in the x_n-direction is exponentially decaying. The author believes that the previous part can also have the exponential decay property in the x_n-direction, if the initial data is required to satisfy more conditions. From this we can infer that the long time behavior is influenced by the direction of the moving boundary.

Before showing our results, we first give some notations in the paper.

Notation. For any $x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n_+$, we will always denote $\bar{x} = (x_1, \ldots, x_{n-1})$. And denote $\nabla_x = (\partial_{x_1}, \ldots, \partial_{x_{n-1}}, \partial_{x_n})$, $\nabla_{\bar{x}} = (\partial_{x_1}, \ldots, \partial_{x_{n-1}})$. $W^{s,p}(\mathbb{R}^n_+)$ represents the usual Sobolev space and the L^p norm of a function is denoted by $\| \cdot \|_p$. Denote

$$erfc(a) = \frac{2}{\sqrt{\pi}} \int_a^\infty e^{-s^2} ds, \tag{1.4}$$

for any $a \in \mathbb{R}$. For any function $f(t)$ defined in $t \geq 0$, let $\mathfrak{L}[f]$ be the Laplace transform of $f(t)$,

$$\mathfrak{L}[f](s) = \int_0^\infty f(t) e^{-st} dt, \tag{1.5}$$

and $\mathfrak{L}^{-1}[\cdot]$ the inverse Laplace transform. We also denote generic constants by C which may vary line by line according to the context, and c represents a universal constant. Our main results are as follows:

Theorem 1.1 (*Case $l < 0$*). *Suppose $l < 0$. Also suppose that $u_0 \in L^1 \cap H^2 \cap W^{1,\infty}(\mathbb{R}^n_+)$ and*

$$\|u_0\|_1 + \|u_0\|_\infty \leq \varepsilon,$$

for some constant ε depending only on n, β and λ. Then the problem (1.3) is globally solvable in $C^{2,1}(\mathbb{R}^n_+ \times (0, +\infty))$ and it holds

$$\|(u(t), v(t))\|_\infty \leq C\varepsilon(1 + t)^{-n/2}, \quad \textit{for all } t \geq 0,$$

where $C = C(n, \beta, \lambda)$ is independent of l.