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Constant mean curvature spacelike hypersurfaces in Lorentzian warped products and Calabi–Bernstein type problems

Juan A. Aledo [a,*], Alfonso Romero [b], Rafael M. Rubio [c]

[a] Departamento de Matemáticas, E.S.I. Informática, Universidad de Castilla-La Mancha, 02071 Albacete, Spain

[b] Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain

[c] Departamento de Matemáticas, Campus de Rabanales, Universidad de Córdoba, 14071 Córdoba, Spain

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ABSTRACT

In this paper we provide several uniqueness and non-existence results for complete parabolic constant mean curvature spacelike hypersurfaces in Lorentzian warped products under appropriate geometric assumptions. As a consequence of this parametric study, we obtain very general uniqueness and non-existence results for a large family of uniformly elliptic EDP's, so solving the Calabi–Bernstein problem in a wide family of spacetimes.

1. Introduction

The study of spacelike hypersurfaces with constant mean curvature (CMC) in Lorentzian manifolds has attracted the interest of a considerable group of geometers as evidenced by the amount of works that it has generated. This is due not only to its mathematic interest, but also to its relevance in General Relativity; a summary of several reasons justifying its interest can be found in [1]. In particular, hypersurfaces of (non-zero) CMC are particularly suitable for studying the propagation of gravity radiation [2]. Classical papers dealing with uniqueness results on CMC spacelike hypersurfaces are, for instance, [3,4,1]. In [3], Brill and Flaherty considered a spatially closed universe, and proved several uniqueness results on CMC hypersurfaces in the large by assuming that the Ricci curvature of the spacetime satisfies that $\overline{\mathrm{Ric}}(z, z) > 0$ for all timelike vectors z. In [1], this energy condition was relaxed by Marden and Tipler to include, for instance, non-flat vacuum spacetimes. Later, Bartnik proved in [5] very general existence theorems on CMC spacelike hypersurfaces, and claimed that it would be useful to find new satisfactory uniqueness results. More recently, in [6] Alias, Romero and Sanchez proved new uniqueness results for CMC hypersurfaces in the class of spacetimes that they call closed generalized Robertson–Walker

* Corresponding author. Tel.: +34 967667268.
E-mail addresses: juanangel.aledo@uclm.es (J.A. Aledo), aromero@ugr.es (A. Romero), rmrubio@uco.es (R.M. Rubio).

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spacetimes (which includes the spatially closed Robertson–Walker spacetimes), under the Temporal Convergence Condition (TCC). Finally, in [7], Romero, Rubio and Salamanca, have provided some uniqueness results for the maximal case (zero mean curvature) in spatially parabolic generalized Robertson–Walker spacetimes, which are open models whose fiber is a parabolic Riemannian manifold.

In the paradigmatic case of CMC hypersurfaces immersed in the Lorentz–Minkowski space $\mathbb{L}^{n+1}$, $n \geq 2$, there is a great variety of results from different points of view. One of the most celebrated results is the solution to the corresponding Bernstein problem for maximal hypersurfaces, known as Calabi–Bernstein problem in this Lorentzian context, by Calabi ($n \leq 4$) [8], and Cheng and Yau (arbitrary n) [9]. As for the case of non-zero constant mean curvature, many nonlinear examples of complete spacelike hypersurfaces with non-zero constant mean curvature can be constructed (see for instance [10–12]). In [13] the spacelike hyperplanes in $\mathbb{L}^{n+1}$ are characterized as the only complete CMC spacelike hypersurfaces which are bounded between two parallel spacelike hyperplanes. On the other hand, Aiyama [14] and Xin [15] simultaneously and independently characterized spacelike hyperplanes as the only complete CMC spacelike hypersurfaces in $\mathbb{L}^{n+1}$ whose image under the Gauss map is bounded in the hyperbolic n-space (see also [16] for a weaker first version of this result given by Palmer). Recall that the Gauss application N of a spacelike hypersurface M immersed in $\mathbb{L}^{n+1}$ can be thought as an application from M into the hyperbolic space $\mathbb{H}^n \subset \mathbb{L}^{n+1}$. Thus, the Gauss application is bounded if and only if the hyperbolic angle between N and the oriented timelike axis is also bounded.

In the general case of a spacelike hypersurface in a Lorentzian manifold $\overline{M}$, the Gauss map of M can be globally defined provided $\overline{M}$ is time-orientable. In general, in this context it has not sense talking about bounded Gauss application, but once we choose a unitary timelike vector field globally defined on $\overline{M}$ (compatible with the time-orientation), the notion of hyperbolic angle can be naturally defined (see Section 2 for the details). Thus, the assumption of bounded hyperbolic angle is the natural extension of the one used by Aiyama and Xin in their result, and actually it has been used in this context [17,18].

In this paper we consider a wide family of Lorentzian manifolds, given by the warped product of an 1-dimensional manifold endowed with a negative definite metric and an n-dimensional ($n \geq 2$) Riemannian manifold which, in general, will be taken complete and non-compact. Note that the classical spatially open Robertson–Walker cosmological models are included in that family.

In these ambient spaces there exists a distinguished unitary timelike vector field globally defined which allows to naturally define the notion of hyperbolic angle for every immersed spacelike hypersurface. Thus, one of our main aims will be to provide characterizations of complete spacelike hypersurfaces with bounded hyperbolic angle under suitable geometric hypothesis (for instance, energy-type conditions) on both the ambient space and the hypersurface. As will be pointed out, the family of warped products for which our results are applicable is very large and contains notable examples.

More precisely, given an $n(\geq 2)$-dimensional (connected) Riemannian manifold (F, g_F), an open interval $I \subseteq \mathbb{R}$ endowed with the metric $-dt^2$ and a positive smooth function f defined on I, the product manifold $I \times F$ endowed with the Lorentzian metric

$$\bar{g} = -\pi_I^*(dt^2) + f(\pi_I)^2\, \pi_F^*(g_F),$$

where π_I and π_F denote the projections onto I and F, respectively, is a *Lorentzian warped product* in the sense of [19, cap. 7]. This kind of Lorentzian manifolds are also known as Generalized Robertson–Walker (GRW) spacetimes in the physical context [6]. Along this paper we will represent this $(n+1)$-dimensional Lorentzian manifold by $\overline{M} = I \times_f F$. When $n = 3$ and the fiber F has constant sectional curvature, $\overline{M} = I \times_f F$ is known as a Robertson–Walker (RW) spacetime. Note that a RW spacetime obeys the *cosmological principle*, i.e. it is spatially homogeneous and spatially isotropic, at least locally. Thus, GRW spacetimes widely extend to RW spacetimes and include, for instance, the Lorentz–Minkowski spacetime, the Einstein–de Sitter spacetime, the Friedmann cosmological models, the static Einstein spacetime and the de Sitter spacetime. GRW spacetimes are useful to analyze if a property of a RW spacetime M is *stable*, i.e. if it remains true for spacetimes close to M in a certain topology defined on a suitable family of spacetimes [20]. In fact, a deformation $s \mapsto g_F^{(s)}$ of the metric of F provides a one parameter family of GRW spacetimes close to M when s approaches to 0. Note that a conformal change of the metric of a GRW spacetime, with a conformal factor which only depends on t, produces a new GRW spacetime. On the other hand, a GRW spacetime is not necessarily spatially homogeneous. Recall that spatial homogeneity seems appropriate just as a rough approach to consider the universe in the large. However, this assumption could not be realistic when the universe is considered in a more accurate scale. Thus, a GRW spacetime could be a suitable spacetime to model a universe with inhomogeneous spacelike geometry [21].

When the fiber F is a compact (without boundary) Riemannian manifold, the Lorentzian warped product $\overline{M} = I \times_f F$ is said to be spatially *closed*. On the other hand, if F is complete and non-compact, we will say that $\overline{M}$ is spatially *open*. In this last case, if moreover F is parabolic then $\overline{M}$ is said to be *spatially parabolic* [7]. The open case is especially interesting since, unlike that in the closed one, it can be compatible with the inflation hypothesis and the holographic principle [22,23]. Moreover, in a spatially open Lorentzian warped product $\overline{M} = I \times_f F$, the boundedness of the hyperbolic angle of a spacelike hypersurface M has a physical interpretation. In fact, consider the unitary normal vector field N on M and the unit timelike vector field $\mathcal{T}_p := -\partial_t$ (the sign minus depends on the chosen time orientation). Along M there exist two families of *instantaneous observers:* $\mathcal{T}_p$, $p \in M$, and the normal observers N_p. The quantities

$$\cosh\varphi(p) \quad \text{and} \quad v(p) := \left(\frac{1}{\cosh\varphi(p)}\right) N_p^F,$$