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On a maximal L_p–L_q approach to the compressible viscous fluid flow with slip boundary condition

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ABSTRACT

In this paper, we prove a local in time unique existence theorem for the compressible viscous fluids in the general domain with slip boundary condition. For the purpose, we use the contraction mapping principle based on the maximal L_p–L_q regularity by means of the Weis operator valued Fourier multiplier theorem for the corresponding time dependent problem. To obtain the maximal L_p–L_q regularity, we prove the sectorial $\mathcal{R}$-boundedness of the solution operator to the generalized Stokes equations.

1. Introduction

We consider the motion of a compressible barotropic viscous fluid occupying a domain Ω of the N dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$) with the boundary slip condition. Let $\rho = \rho(x,t)$ be the density of the fluid, $\mathbf{v} = \mathbf{v}(x,t) = (v_1(x,t), \ldots, v_N(x,t))$ the velocity vector field, and $P(\rho)$ the pressure function with $x = (x_1, \ldots, x_N) \in \Omega$ and t being the time variable. The motion is described by the following equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 & \text{in } \Omega \times (0,T), \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \operatorname{Div} \mathbf{S}(\mathbf{u}) + \nabla P(\rho) = 0 & \text{in } \Omega \times (0,T), \\ \mathbf{D}(\mathbf{u})\mathbf{n} - \langle \mathbf{D}(\mathbf{u})\mathbf{n}, \mathbf{n} \rangle \mathbf{n} = 0, \ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma \times (0,T), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_* + \theta_0, \mathbf{u}_0) & \text{in } \Omega \end{cases} \tag{1.1}$$

(cf. [1,2]), where $\partial_t = \partial/\partial t$, ρ_* is a positive constant describing the mass density of the reference body Ω, Γ the boundary of Ω and $\mathbf{n}$ the unit outward normal to Γ. Moreover, $P(\rho)$ is a C^∞ function defined on $\rho > 0$ satisfying the condition: $P'(\rho) > 0$ for $\rho > 0$ and $\mathbf{S}(\mathbf{u})$ the stress tensor of the form:

$$\mathbf{S}(\mathbf{u}) = \alpha \mathbf{D}(\mathbf{u}) + (\beta - \alpha) \operatorname{div} \mathbf{u} \mathbf{I},$$

where α and β are positive constants describing the first and second viscosity coefficients, respectively, $\mathbf{D}(\mathbf{v})$ denotes the deformation tensor whose (j,k) components are $D_{jk}(\mathbf{v}) = \partial_j v_k + \partial_k v_j$ with $\partial_j = \partial/\partial x_j$, and $\mathbf{I}$ the $N \times N$ identity matrix.

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Finally, for any matrix field $\mathbf{K}$ with components K_{jk}, $j, k = 1, \ldots, N$, the quantity $\mathrm{Div}\, K$ is an N-vector with jth component $\sum_{k=1}^N \partial_k K_{jk}$, and also for any vector of functions $\mathbf{u} = (u_1, \ldots, u_N)$ we set $\mathrm{div}\, \mathbf{u} = \sum_{j=1}^N$ and $\mathbf{u} \cdot \nabla = \sum_{j=1}^N u_j \partial_j$ with $\nabla = (\partial_1, \ldots, \partial_N)$.

A local in time unique existence theorem was proved by Burnat and Zajączkowski [3] in a bounded domain of 3-dimensional Euclidean space $\mathbb{R}^3$, where their velocity field $\mathbf{u}$ and density of the fluid ρ belong to Sobolev–Slobodeckij spaces $W_2^{2+\alpha,1+\alpha/2}$ and $W_2^{1+\alpha,1/2+\alpha/2}$ with $\alpha \in (1/2, 1)$, respectively. Moreover, a global in time unique existence theorem was proved by Kobayashi and Zajączkowski [4] in the same class as in the local in time unique existence theorem by the energy method. The purpose of this paper is to prove a local in time unique existence theorem in a uniform $W_q^{3-1/q}$[1] and our velocity field $\mathbf{u}$ and density of the fluid ρ belong to $W_{q,p}^{2,1}(\Omega \times (0, T))$ and $W_{q,p}^{1,1}(\Omega \times (0, T))$ with $2 < p < \infty$ and $N < q < \infty$, where we have set

$$W_{q,p}^{\ell,m}(\Omega \times (0, T)) = L_p((0, T), W_q^{\ell}(\Omega)) \cap W_p^m((0, T), L_q(\Omega)). \tag{1.2}$$

One of the merits of our approach is less compatibility condition than [3].

To obtain such a local in time unique existence theorem, it is key to prove the L_p–L_q maximal regularity for the linearized problem of the following form:

$$\begin{cases} \partial_t \rho + \gamma_2 \mathrm{div}\, \mathbf{u} = f & \text{in } \Omega \times (0, \infty), \\ \gamma_0 \partial_t \mathbf{u} - \mathrm{Div}\, \mathbf{S}(\mathbf{u}) + \nabla(\gamma_1 \rho) = \mathbf{g} & \text{in } \Omega \times (0, \infty), \\ \alpha[\mathbf{D}(u)\mathbf{u} - \langle \mathbf{D}(\mathbf{u})\mathbf{n}, \mathbf{n}\rangle \mathbf{n}] = \mathbf{h} - \langle \mathbf{h}, \mathbf{n}\rangle \mathbf{n}, \ \mathbf{u} \cdot \mathbf{n} = \tilde{h} & \text{on } \Gamma \times (0, \infty), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega. \end{cases} \tag{1.3}$$

$\gamma_i = \gamma_i(x)$ $(i = 0, 1, 2)$ are uniformly continuous functions defined on $\bar{\Omega}$ satisfying the assumptions:

$$\rho_*/2 \le \gamma_0(x) \le 2\rho_*, \qquad 0 \le \gamma_k(x) \le \rho_1 \quad (x \in \Omega, k = 1, 2), \qquad \|\nabla \gamma_\ell\|_{L_r(\Omega)} \le \rho_1 \quad (\ell = 0, 1, 2) \tag{1.4}$$

with some positive constant ρ_1.

The maximal regularity result was first proved by Solonnikov [5] for the general parabolic equations satisfying the uniform Lopatinski–Shapiro conditions. In his paper, the problem in a domain is transformed locally to the model problems in a neighbourhood of either an interior point or a boundary point by using the localization technique and the partition of unity associated with the domain Ω. The boundary neighbourhood problem (1.3) is transformed to a problem in the half-space $x_N > 0$. By applying the Fourier transform with respect to time and tangential directions, his problem becomes a system of ordinary differential equations. Solonnikov calculated explicitly the inverse Fourier transform of solutions of such ordinary differential equations and expressed them in the form of potentials in the half-space. Then, he estimated them in suitable norms. Burnat and Zajączkowski [3] also used the same procedure as in Solonnikov [5] to transform problem (1.3) to the half-space problem and they estimated the inverse Fourier transform of solutions of the ordinary differential equations by the Plancherel theorem because they worked in the L_2 framework. Kakizawa [6] proved $\mathcal{R}$-boundedness of solutions to the ordinary differential equations corresponding to (1.3) and used the Weis operator valued Fourier multiplier theorem [7] to prove the maximal L_p–L_q regularity in the half-space. The idea in [6] is similar to that in Shibata and Shimizu [8].

Our approach is completely different from these papers [5,3,6], that is we prove the existence of an $\mathcal{R}$ bounded solution operator to the following generalized resolvent problem corresponding to time dependent problem (1.3):

$$\begin{cases} \lambda\theta + \gamma_2 \mathrm{div}\, \mathbf{v} = f & \text{in } \Omega, \\ \gamma_0 \lambda \mathbf{v} - \mathrm{Div}\, \mathbf{S}(\mathbf{v}) + \nabla(\gamma_1 \theta) = \mathbf{g} & \text{in } \Omega, \\ \alpha[\mathbf{D}(\mathbf{v})\mathbf{n} - \langle \mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n}\rangle \mathbf{n}] = \mathbf{h} - \langle \mathbf{h}, \mathbf{n}\rangle \mathbf{n}, \ \mathbf{v} \cdot \mathbf{n} = \tilde{h} & \text{on } \Gamma. \end{cases} \tag{1.5}$$

In fact, we prove that for any $\epsilon \in (0, \pi/2)$, there exist a constant $\lambda_0 \ge 1$ and an operator family $R(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), W_q^2(\Omega)^N))$ such that for any $f \in W_q^1(\Omega)$, $\mathbf{g} \in L_q(\Omega)^N$, $\mathbf{h} \in W_q^1(\Omega)^N$ and $\tilde{h} \in W_q^2(\Omega)$, problem (1.5) admits a unique solution $(\rho, \mathbf{v}) = R(\lambda)(\mathbf{f}, \mathbf{g}, \lambda^{1/2}\mathbf{h}, \nabla \mathbf{h}, \lambda\tilde{h}, \lambda^{1/2}\nabla\tilde{h}, \nabla^2\tilde{h})$ and $(\lambda, \lambda^{1/2}\nabla P_v, \nabla^2 P_v)R(\lambda)$ is $\mathcal{R}$-bounded for $\lambda \in \Sigma_{\epsilon,\lambda_0} \cap K_\epsilon$ with value in $\mathcal{L}(\mathcal{X}_q(\Omega), W_q^1(\Omega) \times L_q(\Omega)^{\tilde{N}})$. Here, P_v is the projection such that $P_v(\rho, \mathbf{u}) = \mathbf{u}$, $\tilde{N} = N + N^2 + N^3$,

$$\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \mid |\lambda| \ge \lambda_0, |\arg \lambda| \le \pi - \epsilon\},$$

$$K_\epsilon = \left\{\lambda \in \mathbb{C} \mid \left(\lambda + \frac{\gamma}{\alpha + \beta} + \epsilon\right)^2 + (\mathrm{Im}\, \lambda)^2 \ge \left(\frac{\gamma}{\alpha + \beta} + \epsilon\right)^2\right\}, \tag{1.6}$$

$$\mathcal{X}_q(\Omega) = \{F = (F_1, \ldots, F_7) \mid F_1 \in W_q^1(\Omega), F_5 \in L_q(\Omega), F_2, F_3, F_6 \in L_q(\Omega)^N, F_4, F_7 \in L_q(\Omega)^{N^2}\},$$

with $\gamma = \sup_{x \in \overline{\Omega}} \gamma_1(x)\gamma_2(x)$, and F_1, F_2, F_3, F_4, F_5, F_6 and F_7 are independent variables corresponding to f, $\mathbf{g}$, $\lambda^{1/2}\mathbf{h}\nabla\mathbf{h}$, $\lambda\tilde{h}$, $\lambda^{1/2}\nabla\tilde{h}$ and $\nabla^2\tilde{h}$, respectively. Moreover, $\mathrm{Hol}\,(U, \mathcal{L}(X, Y))$ denotes the set of all $\mathcal{L}(X, Y)$ valued holomorphic functions

[1] The definition of $W_q^{3-1/q}$ domain is given in Definition 1.1, below.