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Quantitative image recovery theorems

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ABSTRACT

This paper provides a quantitative version of the classical image recovery problem to find an ϵ-approximate solution of the problem. The rate of asymptotic regularity of the iteration schemas, connected with the problem of image recovery, coincides with the existing optimal and quadratic bound for Krasnoselskii–Mann iterations. We then provide explicit effective and uniform bounds on the approximate fixed points of the mappings under consideration to be an approximate solution of the image recovery problem up to a uniform change from ϵ to $\delta_{(\epsilon)}$. When combined, these results provide algorithms with explicit rates of convergence for the recovery of an ϵ-perturbation of the original image in different settings.

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1. Introduction and preliminaries

In 1965, Browder–Göhde–Kirk proposed, independently, the theory of nonexpansive mappings in uniformly convex Banach spaces (see e.g. [1,2]). Since then, the fixed point theory for nonexpansive mappings is an active area of research in nonlinear functional analysis and found a diverse range of applications, for instance problems of zeros of a monotone operator and variational inequality problems. The problem of finding a common fixed point of a finite family of nonlinear mappings acting on a nonempty convex domain often arises in applied mathematics. For example, finding a common fixed point of a finite family of nonexpansive mappings may be used to solve systems of simultaneous equations, convex minimization problems of functions and the problem of image recovery. The latter problem has been analyzed in Hilbert spaces and further generalized to uniformly convex Banach spaces and found many useful applications in applied mathematics, for instance partial differential equations, control theory and image and signal reconstruction. The purpose of this paper is to analyze different iteration schemas, involving a finite family of nonlinear mappings, which are closely related to the problem of image recovery.

Let H be a real Hilbert space and let $C_1, C_2, \dots, C_r$ be nonempty closed convex subsets of H. The problem of image recovery in a real Hilbert space H is defined as follows:

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The original (unknown) image z is known a priori to belong to the intersection $C_0 = \cap_{i=1}^r C_i$ of the closed convex sets $C_1, C_2, \ldots, C_r$. An iteration schema involving the metric projections $P_i : H \to C_i$ onto the corresponding sets C_i, in which some initial estimate is sequentially projected onto the individual sets according to a periodic schedule, recovers some $z \in C_0$. The image recovery problem is also studied under the label 'convex feasibility problem' (see e.g. [3]).

One of the most important notions in metric fixed point theory is the asymptotic regularity [4] of a nonlinear iteration $\{x_n\}$ under consideration (see e.g. [5]). A (Picard iteration of the) nonlinear mapping $T : C \to C$ is said to be asymptotically regular if

$$\lim_{n\to\infty} \left\| T^n x - T^{n+1} x \right\| = 0. \tag{1.1}$$

Asymptotic regularity is not only useful in proving that fixed points exist but also in showing that the sequence of iterates $\{x_n\}$ converges (at least weakly) to a fixed point.

In the theory of image recovery the following form of asymptotic regularity is essentially due to Crombez [6]:

Theorem 1.1 ([6, Theorem 2]). *Let $T : H \to H$ be a mapping given by*

$$T = \alpha_0 I + \sum_{i=1}^r \alpha_i T_i, \quad 0 < \alpha_i < 1 \, \forall \, i = 0, 1, 2, \ldots, r, \qquad \sum_{i=0}^r \alpha_i = 1 \tag{1.2}$$

where

(i) *each T_i is nonexpansive on H;*
(ii) *the set of fixed points of T is nonempty;*
(iii) *$Tu = u \Leftrightarrow T_i u = u \, \forall \, i = 0, 1, 2, \ldots, r$.*

Then T is asymptotically regular.

Conditions (ii) and (iii) can be summarized as $F(T) = \cap_{i=1}^r F(T_i) \neq \emptyset$. However, an inspection of the proof in [6] shows that actually only $\cap_{i=1}^r F(T_i) \neq \emptyset$ is needed for the asymptotic regularity.

This result has been generalized to uniformly convex Banach spaces:

Theorem 1.2 ([7, Theorem 5.4.2]). *Let X be a uniformly convex Banach space with modulus of uniform convexity η and let C be a nonempty convex subset of X. Let $T : C \to C$ be a mapping as defined in (1.2) where each T_i is nonexpansive on C and $\cap_{i=1}^r F(T_i) \neq \emptyset$. Then T is asymptotically regular.*

In the context of the image recovery problem, the mappings T_i are defined as

$$T_i := I + \lambda_i (P_i - I), \tag{1.3}$$

where the $P_i : C \to C_i$ are

(i) metric projections in the case of Hilbert spaces H, $C := H$ and $0 < \lambda_i < 2$ for all i,
(ii) nonexpansive retractions in uniformly convex spaces, where then, however, one has to restrict the coefficients to $0 < \lambda_i < 1$ for all i (nonexpansive retracts and retractions are e.g. discussed in [8–10]).

In 1992, Crombez [11] introduced and analyzed another parallel computing iteration schema by considering a mapping T as a convex combination solely of the mappings T_i defined in (1.3). That is

$$T = \sum_{i=1}^r \alpha_i T_i, \quad \forall \, i = 1, 2, \ldots, r; \; \alpha_i > 0 \text{ and } \sum_{i=1}^r \alpha_i = 1. \tag{1.4}$$

Iterates of such mappings T have been studied first in [12]. The Picard iteration of the above mapping T is asymptotically regular and exhibits weak convergence in Hilbert spaces.

Metric projections are an essential ingredient of the iteration schema used for the recovery of the image. Since the construction of T_i involves the metric projection P_i which is characterized as a nonexpansive mapping in Hilbert spaces also the relaxed metric projection T_i defined in (1.3) is nonexpansive. Furthermore, the set of common fixed points of P_i coincides with that of T_i. Crombez [6,11] shows that the set of fixed points of T coincides with $C_0 = \cap_{i=1}^r C_i$, where T is defined as in (1.2) or as in (1.4). From this he gets that $(T^n(x))$ weakly converges to a point $p \in C_0 = \cap_{i=1}^r C_i = \cap_{i=1}^r F(T_i)$.

However, the classical image recovery problem lacks any information on how a δ-fixed point of T relates to being in the intersection $C_{0,\epsilon}$ of ϵ-neighborhoods $C_{i,\epsilon}$ of C_i. Moreover, the problem of image recovery is often and seriously dealt with the inconsistent constraints i.e., when the intersection of the sets $C_1, C_2, \ldots, C_r$ is empty (see e.g. [13,14]). To answer this question but also to get explicit effective rates of convergence we now introduce an ϵ-version of the classical image recovery problem which we will solve with explicit bounds in this paper. Our proposed ϵ-version provides an approximate solution of the problem even in a situation when constraints are inconsistent.

Let H be a real Hilbert space and let $C_1, C_2, \ldots, C_r$ be nonempty closed convex subsets of H. Let $\epsilon > 0$ and let $C_{1,\epsilon}, C_{2,\epsilon}, \ldots, C_{r,\epsilon}$ be the corresponding 'r' nonempty ϵ-convex subsets of H, where $C_{i,\epsilon} := \cup_{x \in C_i} B_\epsilon(x)$ (for $1 \leq i \leq r$)

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