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Ergodicity of the stochastic fractional reaction–diffusion equation[☆]

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ABSTRACT

A stochastic reaction–diffusion equation with fractional dissipation is considered. The smaller the exponent of the equation is, the weaker the dissipation of the equation is. The equation is discussed in detail when the exponent changes. The aim is to prove the well-posedness, existence and uniqueness of an invariant measure as well as strong law of large numbers and convergence to equilibrium. Without the analytic property of the semigroup, some methods are used to overcome the difficulties to get the energy estimates. The results in this paper can be applied to the classic reaction–diffusion equation with Wiener noise.

1. Introduction

As well known, nowadays, it has been universally acknowledged in the physical, chemical and biological communities that the reaction–diffusion equation plays an important role in dissipative dynamical systems. Typical examples are provided by the fact that there are many phenomena in biology where a key element or precursor of a developmental process seems to be the appearance of a traveling wave of chemical concentration (or mechanical deformation). When reaction kinetics and diffusion are coupled, traveling waves of chemical concentration can effect a biochemical change much faster than straight diffusional processes. This usually gives rise to reaction–diffusion equations as follows

$$\frac{\partial u}{\partial t} + \nu \triangle u = f(u), \tag{1.1}$$

where u denotes the chemical concentration, ν is the diffusion coefficient and the function $f(u)$ represents the kinetics. If $f(u)$ is linear, i.e., $f(u) = k_1 u + k_2$, where k_1 and k_2 are real constants, then Eq. (1.1) can be solved by the separation of variables methods. If $f(u)$ is nonlinear, then the problem is much more intractable. The classic and simplest case of the nonlinear reaction–diffusion equation is the so-called Fisher equation, as the function $f(u) = (k_4 u^2 - k_3 u)$, which was suggested by Fisher as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population [1]. In the 20th century, the Fisher equation has become the basis for a variety of models for spatial spread. The typical examples are that

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Aoki discussed gene-culture waves of advance [2] and Ammerman and Cavalli-Sforza, in an interesting direct application of the model, applied it to the spread of early farming in Europe [3,4]. Meanwhile, the solution of the Fisher equation has been widely investigated in [5–9]. If the nonlinear term $f(u) = k_7u^3 + k_6u^2 + k_5u$, where k_5, k_6 and k_7 are real constants, Eq. (1.1) can be regarded as a generalization of the Fisher equation, which is used as a density-dependent diffusion model, in the one-dimensional situation, for studying insect and animal dispersal with growth dynamics [10], and as a genetic model arising from the classical theory of population genetics and combustion [11,12]. More results of Eq. (1.1) can be found in [13–17] and their numerous references.

Stochastic partial differential equations play an important role in the mathematical modeling of many physical phenomena. These equations not only generalize the models of the deterministic cases, but they lead to new phenomena which is important in physics. For example, Crauel and Flandoli [18] showed that the deterministic pitchfork bifurcation disappears as soon as an additive white noise of arbitrarily small intensity is incorporated in the model. Hairer and Mattingly [19] characterized the class of noises for which the 2 dimensional stochastic Navier–Stokes equation is ergodic. In recent series of papers and lectures, Flandoli et. al. proved that for several examples of deterministic partial differential equations which are illposed a suitable random noise can restore the illposedness, see e.g. [20–22]. C. Marinelli and M. Rökner [23] studied the ergodicity for stochastic reaction–diffusion equations with multiplicative Poisson noise. Peter W. Bates, Kening Lu and Bixiang Wang [24] proved the existence of the random attractors for stochastic reaction–diffusion equations on unbounded domains. And there are lots of other work, see for short list e.g. [25–27] and references therein. But the stochastic reaction–diffusion equations do not recover the stochastic fractional reaction–diffusion equation.

More precisely, we consider the stochastic fractional reaction–diffusion equation in a bounded domain with Wiener noise as the body forces like this

$$\begin{aligned} &du + \left((-\Delta)^{\frac{\alpha}{2}} u + u^3 - u\right) dt = dW, \quad t > 0,\ x \in (0, 1), \\ &u(t, 0) = u(t, 1) = 0, \quad t > 0, \\ &u(0, x) = u_0(x), \quad x \in [0, 1], \end{aligned} \tag{1.2}$$

where $\alpha \in (1, 2)$, W stands for an $L^2(0, 1)$-cylindrical Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with expectation E and normal filtration $\mathcal{F}_t = \sigma\{W(s) : s \le t\}$, $t \in [0, T]$. And it has the following representation

$$W(t) = \sum_{k=1}^{\infty} e_k \beta_k(t), \quad t \in [0, T],$$

where e_k, $k \in \mathbb{N}$, is an orthonormal basis of $L^2(0, 1)$ and β_k, $k \in \mathbb{N}$ is a family of independent real-valued Brownian motions.

The fractions of the Laplacian are the infinitesimal generators of Lèvy stable diffusion processes and appear in anomalous diffusions in plasma, flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics, and American options in finance. The equations with fractional diffusion are becoming popular in many areas of applications [28,29]. In these applications, it is often important to consider boundary value problems. Hence it is useful to study solutions for space fractional diffusion equations on bounded domains with Dirichlet boundary conditions.

Recently, the authors notice that papers [30,31] are relative to our work. In the two papers, Z. Brzezniak, L. Debbi and Ben Goldys consider the case of multiple noise for the fractional Burgers equation. They use Itö's formula, Burkholder–Davis–Gundy inequality, Girsanov theorem and so on techniques in stochastic analysis to prove the global well posedness and ergodicity in $L^2(0, 1)$ when the exponent $\alpha \in (\frac{3}{2}, 2)$. In the present work, we study the case of additive noise for fractional reaction–diffusion equations. We use the contraction principle to consider the local existence of the solution $u(t)$ in $L^2(0, 1)$, but it produces $\|u(t)\|_{L^3}$ which cannot be dominated by $\|u(t)\|_{L^2}$. This is different from [30]. So we study the equation in H^σ which satisfies $H^\sigma \subset L^3(0, 1)$ with $\sigma \in [\frac{1}{6}, \frac{\alpha}{2}]$. This will bring more difficulties. Here we relax the assumption of $\alpha \in (\frac{3}{2}, 2)$ in [30] to $\alpha \in (1, 2)$. By changing the stochastic equations into random partial differential equations (PDE), we use techniques from PDE to obtain the global well-posedness in $C([0, T]; H^\sigma)$ for $\sigma \in [\frac{1}{6}, \frac{\alpha}{2}]$ as well as ergodicity in $H^{\frac{\alpha}{2}}$. Furthermore, if $\alpha \in (\frac{3}{2}, 2)$, we prove the ergodicity in H^σ with arbitrary $\sigma \in [\frac{1}{6}, \frac{\alpha}{2}]$.

The remainder of this paper is organized as follows. In Section 2, we introduce some notations, give the definition of a mild solution to (1.2), and then show the local existence to (1.2) in H^σ with $\sigma \in [\frac{1}{6}, \alpha - \frac{1}{2})$. In Section 3, by the priori estimates, we get the global well-posedness of the solution to (1.2) in H^σ with $\sigma \in [\frac{1}{6}, \frac{\alpha}{2}]$. In Section 4, using the Krylov–Bogoliubov method we obtain the existence of the invariant measures to the stochastic fractional reaction–diffusion equation in H^σ with $\sigma \in [\frac{1}{6}, \frac{\alpha}{2}]$. In Section 5, by checking irreducibility property and strong Feller property of the Markov semigroup corresponding to the solution of (1.2) we establish the ergodicity of invariant measures in $H^{\frac{\alpha}{2}}$. Finally, we give a remark that if $\alpha > \frac{3}{2}$, we can impose some appropriate assumptions on $W(t)$ to prove the ergodicity for problem (1.2) in H^σ for $\sigma \in [\frac{1}{6}, \frac{\alpha}{2}]$. As usual, constants C may change from one line to the next, unless, we give a special declaration; we denote by $C(a)$ a constant which depends on some parameter a.