Nonlinear Analysis 109 (2014) 23–32

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Instability of plane shear flows

Jiangang Qi, Shaozhu Chen, Bing Xie *

Department of Mathematics, Shandong University, Weihai, Weihai 264209, PR China

ARTICLE INFO

Article history:
Received 10 January 2014
Accepted 30 June 2014
Communicated by Enzo Mitidieri

MSC:
76E20
34B24
34L15

Keywords:
Rayleigh equation
Instability
Shear flow
Sturm–Liouville problem

ABSTRACT

This paper obtains the existence of unstable modes of the Rayleigh equation with smooth or piecewise smooth steady states. In the reasoning, we employ a crucial tool that displays the asymptotic distribution of eigenvalues of non-symmetric Sturm–Liouville operators. For flows with piecewise smooth velocity profiles, we propose a new framework for eigenvalue problems with interior singularities. Examples are presented to show the scope of our criteria.

1. Introduction

In this paper, we consider the hydrodynamic instability problem for plane shear flows. The purpose is to get sufficient conditions for linear instability. For plane shear flows, this problem has a long history going back to scientists such as Lord Rayleigh and Lord Kelvin in the nineteenth century. Consider a parallel plane shear flow $\mathbf{U} = \mathbf{i}U(y)$ in the x-direction within the channel $x \in (-\infty, \infty)$ and $y \in [-1, 1]$. The linearized vorticity equation for a two-dimensional disturbance [1–3] is

$$\partial_t \omega + U(y)\partial_x w - U''(y)\partial_x \psi = 0, \tag{1.1}$$

where $\omega = \omega(x, y, t)$ is the vorticity perturbation, $\psi = \psi(x, y, t)$ is the stream function associated to ω by

$$\omega = \nabla^2 \psi = \frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi,$$

subject to the boundary condition $\psi(\pm 1, t) = 0$.

By the normal mode method, seeking solutions in the form

$$\psi(x, y, t) = \varphi(y)e^{i\alpha(x-ct)}$$

with α the wave number (positive real) in the x-direction and $c = c_r + ic_i$ the complex wave speed, we obtain the Rayleigh equation

$$(U(y) - c)\left(\varphi'' - \alpha^2\varphi\right) - U''(y)\varphi = 0, \quad y \in [-1, 1] \tag{1.2}$$

* Corresponding author. Tel.: +86 6315688630.
E-mail addresses: qjg816@163.com (J. Qi), szchen@sdu.edu.cn (S. Chen), xiebing@sdu.edu.cn (B. Xie).

http://dx.doi.org/10.1016/j.na.2014.06.010

with the boundary condition

$$\varphi(-1) = 0, \qquad \varphi(1) = 0. \tag{1.3}$$

Then the instability problem for shear flows is reduced to the Rayleigh eigenvalue problem, that is, the flow is linearly unstable if there exists a nontrivial solution to (1.2) and (1.3) with $c_i > 0$. A classical result of Lord Rayleigh [3] is the necessary condition for instability that the basic velocity profile have an inflection point $y = y_s$, i.e., $U''(y_s) = 0$. This condition was later improved by Fjørtoft [4].

In 1935, Tollmien [5] obtained an unstable solution to (3) by formally perturbing around a neutral mode (c real) for symmetric flows. M.N. Rosenbluth and A. Simon [6] obtained a necessary and sufficient condition for instability of a class of monotone flows. Recently, some instability criteria have been obtained for the special flows $U(y) = \sin my$ in [7] and $U(y) = \cos my$ in [8]. These results were extensively improved and extended by Z. Lin to more general odd symmetric flows in [9] and other classes of shear flows in [10].

Proposition 1.1. *[cf. [9, Theorem 1.4]] Assume that $U \in C^2[-1, 1]$ is odd and $K(y) := U''(y)/U(y)$ is bounded. Let H_0 be the operator generated by $-\mathrm{d}^2/\mathrm{d}y^2 + U''(y)/U(y)$ with the boundary condition (1.3). If H_0 has a negative eigenvalue, then there is a non-trivial solution to the Rayleigh equation (1.2) with $c = \mathrm{i}\lambda_0$ (here $\lambda_0 > 0$), for a wave number in some areas. Specifically, if $\lambda_1, \lambda_2, \ldots, \lambda_N$ are all the negative eigenvalues of H_0, $\lambda_1 < \lambda_2 < \cdots < \lambda_N < 0$, $\alpha_j = \sqrt{-\lambda_j}$, $j = 1, \ldots, N$, then there is a purely growing instability for every wave number α belonging to the set*

$$(\alpha_N, \alpha_{N-1}) \cup \cdots \cup (\alpha_{2k}, \alpha_{2k-1}) \cup \cdots \cup (\alpha_2, \alpha_1) \text{ for even } N,$$

or to the set

$$(\alpha_{N-1}, \alpha_{N-2}) \cup \cdots \cup (\alpha_{2k}, \alpha_{2k-1}) \cup \cdots \cup (\alpha_2, \alpha_1) \text{ for odd } N.$$

In the first result of this paper, we allow the function $K(y)$ in a wider class $L^1[-1, 1]$ and, in particular, $K(y)$ may be unbounded.

Theorem 1.2. *Suppose that $U(y)$ is odd symmetric on $(-1, 1)$, $U \in C^2[-1, 1]$ and $K = U''/U \in L^1[-1, 1]$. If $-\mathrm{d}^2/\mathrm{d}y^2 + K(y)$ with the boundary condition (1.3) has negative eigenvalues*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N < 0,$$

then there is an unstable mode for every wave number $\alpha \in \bigcup_{k=1}^{m_0} (\alpha_{2k}, \alpha_{2k-1})$, where $m_0 = [(N + 1)/2]$, the largest integer less than or equal to $(N + 1)/2$, $\alpha_j = \sqrt{-\lambda_j}$ for $1 \le j \le N$ and $\alpha_{N+1} = 0$.

Remark 1. Note that we add an interval $(0, \alpha_N)$ to the set for α when N is odd to patch up a flaw in Proposition 1.1.

In the next two results we will further loosen the restriction on the function $U(y)$, i.e., we will use the following piecewise smoothness assumption under which $U''(y)$ may be unbounded, and hence, $K(y)$ may not be locally integrable.

$$\begin{gathered} U(y) \text{ is odd, } U \in C[-1, 1] \text{ and there exist } 2N + 1 \text{ points } \{y_j\}_{-N}^{N} \text{ in } [-1, 1] \text{ such that } 0 = y_0 < y_1 < \cdots < \\ y_N = 1, y_{-j} = -y_j \text{ and } U(y) \text{ is twice continuously differentiable for } y \neq y_j, \ -N \le j \le N. \end{gathered} \tag{1.4}$$

Theorem 1.3. *If $U(y)$ satisfies (1.4) and there exists $c \in \mathbb{C}$ with $\mathrm{Im}c > 0$ such that*

$$\int_{-1}^{1} \frac{\mathrm{d}y}{(U(y) - c)^2} = 0, \tag{1.5}$$

then there exists an $\alpha_c > 0$ such that the Rayleigh eigenvalue problem of (1.2) with (1.3) has an unstable mode for every $\alpha \in (0, \alpha_c)$.

Theorem 1.4. *Suppose that (1.4) holds. If there exist constants $C_j > 0$ and $0 \le \rho_j < 1$, $0 \le j \le N$, such that*

$$|U(y)| \ge C_j |y - y_j|^{\rho_j} \quad \text{for } y \text{ near } y_j, \tag{1.6}$$

then there exists an $\alpha_c > 0$ such that the Rayleigh problem of (1.2) with (1.3) has at least one unstable mode for every $\alpha \in (0, \alpha_c)$.

Section 3 will give a proof of Theorem 1.2, in which we are inspired by [9] but employ a different method, namely, by means of the asymptotic properties of eigenvalues of non-symmetric Sturm–Liouville operators to weaken the smoothness condition on $K(y)$. Theorems 1.3 and 1.4 deal with the Rayleigh equation with piecewise smooth velocity profiles whose derivatives may be discontinuous at junctions. An immediate consequence is that $K(y)$ may not be locally integrable and hence the Rayleigh eigenvalue problem may be singular at some interior points. A special case with a piecewise linear velocity profile was studied in [1, p. 299], but there are no general results. We will propose a framework for eigenvalue problems with interior singularities and prove Theorems 1.3 and 1.4 in Section 4. We will show several illustrative examples among which the last will serve as a comparison to the result in [1, p. 299].