Nonlinear Analysis 109 (2014) 45–55

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Non radial solutions for a non homogeneous Hénon equation[☆]

M. Badiale[a], G. Cappa[b,*]

[a] Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy

[b] Dipartimento di Matematica e Informatica, Università di Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy

ARTICLE INFO

Article history:
Received 3 April 2014
Accepted 4 June 2014
Communicated by Enzo Mitidieri

MSC:
35J20
35J60

Keywords:
Hénon equation
Nehari manifold
Semilinear elliptic equation

ABSTRACT

In this paper we study a Hénon-like equation (see Eq. (1)), where the nonlinearity $f(t)$ is not homogeneous (i.e., it is not a power). By minimization on the Nehari manifold, we prove that for large values of the parameter α there is a breaking of symmetry and non radial solutions appear. This holds for sub- and super-critical growth of the nonlinearity f.

1. Introduction

In this paper we study the following Hénon-like equation

$$\begin{cases} -\Delta u = |x|^{\alpha} f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is the unity ball of $\mathbb{R}^n$ and $n \geq 4$. From a well known result of Ni [1] it derives, assuming suitable hypotheses on f, that (1) has a radial solution. In the case in which $f(t)$ is a power, say $f(t) = |t|^{p-2}t$, the problem is known as Hénon's equation (see [2]). A seminal paper of Smets, Su and Willem [3] showed that, for large α's, there is a breaking of symmetry and a new, non radial, solution appears. After that, much work has been made to study these non radial solutions: multiplicity, shape, and asymptotic behavior. To have just an idea of the research on this topic, one can see for example [4–6] for results about the critical and supercritical cases, [7,8] for the study of the asymptotic behavior of the maximum point and the existence of multi-peak solutions, [9] for the uniqueness of the radial solution for $2 < p < \frac{2n+2\alpha}{n-2}$, [10,11] for results about the p-Laplacian, and [12] for the study of Hénon type system. See also the references in the quoted papers.

At the best of our knowledge, all papers on non radial solutions of the Hénon equation deal with the case in which the nonlinearity is a power.

In this paper we prove a result of existence of a non radial solution, for large α's, in the case in which f is not a power (but not too different from a power). Borrowing some ideas and some results from [6], we also prove existence of non radial solutions for a range of growth of f including supercritical growth. We will find the solutions as minima on the Nehari manifold of the functional usually associated to (1). So the main points of the present paper can be summarized as follows: non homogeneous nonlinearity, supercritical growth, and Nehari manifold.

☆ This research was partially supported by the PRIN2012 grant "Aspetti variazionali e perturbativi nei problemi differenziali nonlineari".

* Corresponding author. Tel.: +39 0521906968.
E-mail addresses: marino.badiale@unito.it (M. Badiale), gianluca.cappa@nemo.unipr.it (G. Cappa).

http://dx.doi.org/10.1016/j.na.2014.06.020
0362-546X/

To write down our result, we first define $l = n/2$ and $p^*(n) = 2\frac{n+2}{n-2}$ if n is even, $l = [n/2] + 1$ and $p^*(n) = 2\frac{[n/2]+2}{[n/2]}$ if n is odd.

We show that such problem admits a radial solution and a non radial solution under the following hypotheses on f:

(f_1) f is a Hölder continuous function (locally), $f(z) \geq 0\ \forall z > 0$, $f(z) = o(z)$ for $z \to 0$; moreover $\lim_{z\to+\infty} \frac{f(z)}{z} = +\infty$, $f(z) = 0$ for all $z \leq 0$;

(f_2) $|f(z)| \leq C(1 + |z|)^{p-1}$, where $2 < p < p^*(n)$ for all z;

(f_3) there exists $q > 2$ such that $qF(t) \leq tf(t)$ for all $t \in \mathbb{R}$, where $F(t) = \int_0^t f(s)ds$.

(f_4) there exist $\mu_1, \mu_2 > 2$ such that for all $t \in [0, 1]$ and $v \geq 0$ we have $f(tv) \geq t^{\mu_1-1} f(v)$ and for all $t \geq 1$ and $v \geq 0$ we have $f(tv) \geq t^{\mu_2-1} g(v)$ where $g(\cdot)$ is a non negative continuous function on $\mathbb{R}$ such that $g(0) = 0$ and with

$$4\frac{\mu_1 - \mu_2}{(\mu_1 - 2)(\mu_2 - 2)} < n - l. \tag{2}$$

Remarks. Some examples of function that satisfy the hypothesis are $f(t) = t^{p-1} + t^{q-1}$ with $p < q$ and $\mu_1 = \mu_2 = q$, $f(t) = \frac{t^q}{1+t^{q-p}}$ or $f(t) = \min\{t^{p-1}, t^{q-1}\}$ with $p = \mu_1$ and $q = \mu_2$ that satisfy (2).

To state our results we introduce the usual Sobolev space $H_0^1(\Omega)$ and its subspace $H_{0,rad}^1(\Omega)$ of radial functions, that is

$$H_{0,rad}^1(\Omega) = \left\{u \in H_0^1(\Omega)\backslash\{0\} : u(x) = u(|x|)\right\}.$$

We then introduce the usual functional associated to problem (1), that is

$$I_\alpha(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega |x|^\alpha F(u)\, dx,$$

and the Nehari manifolds of the functional on $H_{0,rad}^1(\Omega)$:

$$N_{\alpha,r} = \left\{u \in H_{0,rad}^1(\Omega)\backslash\{0\} : \int_\Omega |\nabla u|^2 dx = \int_\Omega |x|^\alpha f(u)u\, dx\right\}.$$

From the results of [1] it easily follows that I_α is a well defined C^1 functional on $H_{0,rad}^1(\Omega)$, for large α's, that is for $\frac{2n+2\alpha}{n-2} \geq p^*(n)$. Also the following theorem is a particular consequence, suitable for our purposes, of the results of [1].

Theorem 1. *Under the hypotheses (f_1), (f_2), (f_3), (f_4), for $\frac{2n+2\alpha}{n-2} \geq p^*(n)$, there is $u \in H_{0,rad}^1(\Omega)\backslash\{0\}$, non-negative solution of (1), that realizes the minimum on the Nehari manifold $N_{\alpha,r}$ that is:*

$$I_\alpha(u_\alpha) = m_{\alpha,r} = \min_{v\in N_{\alpha,r}} I_\alpha(v).$$

In this work we prove the following theorem.

Theorem 2. *Under the hypotheses (f_1), (f_2), (f_3) and (f_4) the problem (1) admits a non radial solution (which is also non negative) for large α's.*

For future use let us notice that from hypothesis (f_4) it follows that

$$\forall t \in (0, 1), \quad v > 0 \text{ results } F(tv) \geq t^{\mu_1} F(v)$$
$$\forall t > 1, \quad v > 0 \text{ results } F(tv) \geq t^{\mu_2} G(v)$$

with $G(v) := \int_0^v g(t)dt$.

The rest of this paper is devoted to the proof of Theorem 2. As usual in the study of Hénon equation, to get the proof we first estimate the "radial critical level" $m_{\alpha,r}$, then we estimate other critical levels and we show that, for large α's, they are distinct.

2. Estimate of $m_{\alpha,r}$

We have defined

$$m_{\alpha,r} = \inf_{u\in N_{\alpha,r}} I_\alpha(u).$$

Thanks to Theorem 1 $m_{\alpha,r}$ is attained, i.e. is a minimum. In this section we get an estimate from below for $m_{\alpha,r}$.