Nonlinear Analysis 109 (2014) 85–102

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Some minimax theorems of set-valued maps and their applications

Issei Kuwano *

Faculty of Engineering, Kanagawa University, Kanagawa 221–8686, Japan

ARTICLE INFO

Article history:
Received 30 September 2013
Accepted 17 June 2014
Communicated by Enzo Mitidieri

MSC:
49J53
54C60
90C46
90C47

Keywords:
Set-valued optimization
Scalarization
Minimax theorem

ABSTRACT

In the paper, we consider two types of scalarization functions of sets and investigate their properties. Moreover, based on two set-relations, we propose two kinds of minimax and maximin values of set-valued maps, respectively, and show some minimax theorems of set-valued maps with respect to those minimax and maximin values by using several properties of the above two functions. As an application of these results, we give common saddle point theorems of vector-valued functions.

1. Introduction

Minimax theorems including saddle point problems of real-valued or vector-valued functions have been investigated by many researchers. Especially, Tanaka [1–3] shows some vector-valued minimax theorems by using scalarization methods of vectors in the objective space. In recent years, some researchers consider a concept of minimax and maximin values of set-valued maps, which is based on a vector criterion, and show some types of minimax theorems for set-valued maps [4–9]. Also, to show their results they use scalarization methods of vectors in the objective space.

On the other hand, many researchers have investigated scalarization as one of the important tools in vector optimization (see [10–12] and references therein). In particular, a sublinear scalarization function of vectors introduced by Rubinov [13] has many useful properties to study nonconvex vector optimization for the weak case (see [11,14] and references therein). Also, some researchers consider certain generalizations of this function and apply them to several problems in set-valued optimization [15–17]. In [18], Hamel and Löhne propose several scalarization functions of sets, which are based on two types of set-relations introduced in [19]. By using these functions, they show generalized results on Ekeland variational principle in an abstract space like topological vector space without such strong assumption as convexity. In [20–23], properties and applications of Hamel and Löhne type scalarization functions are investigated in set-valued optimization. Furthermore, based on the approach of Hamel and Löhne, and six kinds of set-relations introduced in [19], we propose twelve types of scalarization functions of sets in [24]. These functions include all of scalarization functions above, and so we call them unified types of scalarization functions of sets.

* Tel.: +81 45 481 3732.
E-mail address: kuwano@kanagawa-u.ac.jp.

http://dx.doi.org/10.1016/j.na.2014.06.011

In the paper, we consider set-valued minimax theorems including vector-valued minimax theorems with a set-criterion. The motivation of this research is as follows. In general, minimax and maximin values of the vector-valued function f on $X \times Y$ are defined as follows:

$$\min \bigcup_{y\in Y} \max f(x, y) \quad \text{and} \quad \max \bigcup_{x\in X} \min f(x, y).$$

However, are they really minimax and maximin values of f? $F(x) := \max f(x, y)$ and $G(y) := \min f(x, y)$ are generally set-valued maps. Therefore, we think it is more suitable that "min" of $\min \max f$ and "max" of $\max \min f$ are defined by a set criterion.

The aim of the paper is to investigate vector-valued set-valued minimax theorems, which are based on two set-relations introduced in [19], and their applications. For this end, we consider two types of scalarization functions of sets, which are special cases of Hamel and Löhne type functions, and investigate several properties of them.

The organization of the paper is as follows. In Section 2, we introduce some basic concepts in set-valued optimization. In Section 3, we introduce two types of scalarization functions of sets, and investigate their properties. Moreover, we give sufficient conditions for the existence of solutions for certain set-valued optimization problems. In Section 4, we define two types of minimax and maximin values of set-valued maps, respectively, and show some minimax theorems with respect to those values. In Section 5, we consider parametric vector minimax problems and investigate common saddle point theorems of vector-valued functions as an application of set-valued minimax theorems in Section 4.

2. Basic concepts in set-valued optimization

Firstly, we give the preliminary terminology and notation, which will be used in the paper. Let $(Z, \|\cdot\|)$ be the real normed space (Z for short) and $A, B \subset Z$ with $A \neq \emptyset$ and $B \neq \emptyset$. We denote the origin of Z by θ_Z; the family of all nonempty subsets of Z by $\wp(Z)$; the topological interior and complement of A by $\operatorname{int} A$ and A^c, respectively; the product of $\alpha \in \mathbb{R}$ and A by $\alpha A := \{\alpha a | a \in A\}$; the algebraic sum, algebraic difference of A and B by $A + B := \{a + b | a \in A,\ b \in B\}$, $A - B := \{a - b | a \in A,\ b \in B\}$, respectively; the convex hull of A by $\operatorname{conv} A$. Moreover, we denote the set of all non-negative real numbers by $\mathbb{R}_+$; the set of all extended real numbers by $\bar{\mathbb{R}}$, that is, $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Furthermore, we denote the algebraic sum and algebraic difference of a set A and a family of nonempty sets $\mathcal{B}$ by $A + \mathcal{B} := \{A + B : B \in \mathcal{B}\}$ and $A - \mathcal{B} := \{A - B : B \in \mathcal{B}\}$, respectively.

Throughout the paper, X and Y are real Hausdorff topological vector spaces, Z is the real normed space, C is a nontrivial, closed, pointed and convex cone in Z (that is, $C \neq \{\theta_Z\}$, $C \neq Z$, $C + C = C$, $C \cap C = \{\theta_Z\}$ and $\lambda C \subset C$ for all $\lambda \geq 0$) with $\operatorname{int} C \neq \emptyset$. We define a partial ordering $\leq_C$ as follows:

$$x \leq_C y \quad \text{if } y - x \in C \text{ for } x, y \in Z.$$

When $x \leq_C y$ for $x, y \in Z$, we define the order interval between x and y by

$$[x, y]_C := \{z \in Z | x \leq_C z \text{ and } z \leq_C y\}.$$

If $k \in \operatorname{int} C$, then $\operatorname{int}[-k, k]_C = [-k, k]_{\operatorname{int} C}$. When $x, y \in \mathbb{R}$ and $C := \mathbb{R}_+$, $[x, y]_C$ (resp., $\operatorname{int}[x, y]_C$) is denoted by $[x, y]$ (resp., $]x, y[$).

Let $A \in \wp(Z)$. Then $a_0 \in A$ is said to be minimal element of A iff

$$(\{a_0\} - C) \cap A = \{a_0\};$$

maximal element of A iff

$$(\{a_0\} + C) \cap A = \{a_0\}.$$

If C is replaced by $\operatorname{int} C$, then it is called weak minimal element (resp., weak maximal element) of A. We denote the set of all minimal (resp., weak minimal, maximal, weak maximal) elements of A by $\min^{\mathrm{v}} A$ (resp., $\min^{\mathrm{v}}_{\mathrm{w}} A$, $\max^{\mathrm{v}} A$, $\max^{\mathrm{v}}_{\mathrm{w}} A$).

Now we consider two types of set-relation. Let $A_1, A_2 \in \wp(Z)$. Then we write

$$A_1 \leq_C^{(l)} A_2 \quad \text{by } A_2 \subset A_1 + C.$$

$$A_1 \leq_C^{(u)} A_2 \quad \text{by } A_1 \subset A_2 - C.$$

Based on these set-relations, Kuroiwa [25] proposes the following minimal and maximal element concepts of a family of sets. Let $\mathcal{A} \subseteq \wp(Z)$. Then $A_0 \in \mathcal{A}$ is said to be type $(*)$ minimal element of $\mathcal{A}$ iff for any $A \in \mathcal{A}$,

$$A \leq_C^{(*)} A_0 \quad \text{implies } A_0 \leq_C^{(*)} A,$$

and type $(*)$ maximal element of $\mathcal{A}$ iff for any $A \in \mathcal{A}$,

$$A_0 \leq_C^{(*)} A \quad \text{implies } A \leq_C^{(*)} A_0,$$

where $* = l, u$. We denote the family of all type $(*)$ minimal elements (resp., maximal elements) of $\mathcal{A}$ by $\mathrm{Min}^{*}\mathcal{A}$ (resp., $\mathrm{Max}^{*}\mathcal{A}$). Also, we denote the family of all type $(*)$ weak minimal elements (resp., weak maximal element) of $\mathcal{A}$ by $\mathrm{Min}^{*}_{\mathrm{w}}\mathcal{A}$ (resp., $\mathrm{Max}^{*}_{\mathrm{w}}\mathcal{A}$) where $* = l, u$.