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A Hamiltonian vorticity–dilatation formulation of the compressible Euler equations

Mónika Polner[a,*], J.J.W. van der Vegt[b]

[a] Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, 6720 Szeged, Hungary

[b] Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE, Enschede, The Netherlands

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ABSTRACT

Using the Hodge decomposition on bounded domains the compressible Euler equations of gas dynamics are reformulated using a density weighted vorticity and dilatation as primary variables, together with the entropy and density. This formulation is an extension to compressible flows of the well-known vorticity–stream function formulation of the incompressible Euler equations. The Hamiltonian and associated Poisson bracket for this new formulation of the compressible Euler equations are derived and extensive use is made of differential forms to highlight the mathematical structure of the equations. In order to deal with domains with boundaries also the Stokes–Dirac structure and the port-Hamiltonian formulation of the Euler equations in density weighted vorticity and dilatation variables are obtained.

1. Introduction

The dynamics of an inviscid compressible gas is described by the compressible Euler equations and an equation of state. The compressible Euler equations have been extensively used to model many different types of compressible flows, since in many applications the effects of viscosity are small or can be neglected. This has motivated over the years extensive theoretical and numerical studies of the compressible Euler equations. The Euler equations for a compressible, inviscid and non-isentropic gas in a domain $\Omega \subseteq \mathbb{R}^3$ are defined as

$$\rho_t = -\nabla \cdot (\rho u), \tag{1}$$

$$u_t = -u \cdot \nabla u - \frac{1}{\rho}\nabla p, \tag{2}$$

$$s_t = -u \cdot \nabla s, \tag{3}$$

* Corresponding author. Tel.: +36 304707243.
E-mail addresses: polner@math.u-szeged.hu (M. Polner), j.j.w.vandervegt@utwente.nl (J.J.W. van der Vegt).

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with $u = u(x, t) \in \mathbb{R}^3$ the fluid velocity, $\rho = \rho(x, t) \in \mathbb{R}^+$ the mass density and $s(x, t) \in \mathbb{R}$ the entropy of the fluid, which is conserved along streamlines. The spatial coordinates are $x \in \Omega$ and time t and the subscript means differentiation with respect to time. The pressure $p(x, t)$ is given by an equation of state

$$p = \rho^2 \frac{\partial U}{\partial \rho}(\rho, s), \tag{4}$$

where $U(\rho, s)$ is the internal energy function that depends on the density ρ and the entropy s of the fluid. The compressible Euler equations have a rich mathematical structure [1] and can be represented as an infinite dimensional Hamiltonian system [2,3]. Depending on the field of interest, various types of variables have been used to define the Euler equations, e.g. conservative, primitive and entropy variables [1]. The conservative variable formulation is for instance a good starting point for numerical discretizations that can capture flow discontinuities [4], such as shocks and contact waves, whereas the primitive and entropy variables are frequently used in theoretical studies.

In many flows vorticity is, however, the primary variable of interest. Historically, the Kelvin circulation theorem and Helmholtz theorems on vortex filaments have played an important role in describing incompressible flows, in particular the importance of vortical structures. This has motivated the use of vorticity as primary variable in theoretical studies of incompressible flows, see e.g. [5,2], and the development of vortex methods to compute incompressible vortex dominated flows [6].

The use of vorticity as primary variable is, however, not very common for compressible flows. This is partly due to the fact that the equations describing the evolution of vorticity in a compressible flow are considerably more complicated than those for incompressible flows. Nevertheless, vorticity is also very important in many compressible flows. A better insight into the role of vorticity, and also dilatation to account for compressibility effects, is not only of theoretical importance, but also relevant for the development of numerical discretizations that can compute these quantities with high accuracy.

In this article we will present a vorticity–dilatation formulation of the compressible Euler equations. Special attention will be given to the Hamiltonian formulation of the compressible Euler equations in terms of the density weighted vorticity and dilatation variables on domains with boundaries. This formulation is an extension to compressible flows of the well-known vorticity–stream function formulation of the incompressible Euler equations [5,2]. An important theoretical tool in this analysis is the Hodge decomposition on bounded domains [7]. Since bounded domains are crucial in many applications we also consider the Stokes–Dirac structure of the compressible Euler equations. This results in a port-Hamiltonian formulation [8] of the compressible Euler equations in terms of the vorticity–dilatation variables, which clearly identifies the flows and efforts entering and leaving the domain. An important feature of our presentation is that we extensively use the language of differential forms. Apart from being a natural way to describe the underlying mathematical structure it is also important for our long term objective, viz. the derivation of finite element discretizations that preserve the mathematical structure as much as possible. A nice way to achieve this is by using discrete differential forms and exterior calculus, as highlighted in [9–11].

The outline of this article is as follows. In the introductory Section 2 we summarize the main techniques that we will use in our analysis. A crucial element is the use of the Hodge decomposition on bounded domains, which we briefly discuss in Section 2.2. This analysis is based on the concept of Hilbert complexes, which we summarize in Section 2.1. The Hodge Laplacian problem is discussed in Section 2.3. Here we show how to deal with inhomogeneous boundary conditions, which is of great importance for our applications. These results will be used in Section 3 to define via the Hodge decomposition a new set of variables, viz., the density weighted vorticity and dilatation, and to formulate the Euler equations in terms of these new variables. Section 4 deals with the Hamiltonian formulation of the Euler equations using the density weighted vorticity and dilatation, together with the density and entropy, as primary variables. The Poisson bracket for the Euler equations in these variables is derived in Section 5. In order to account for bounded domains we extend the results obtained for the Hamiltonian formulation in Sections 4 and 5 to the port-Hamiltonian framework in Section 6. First, we extend in Section 6.1 the Stokes–Dirac structure for the isentropic compressible Euler equations presented in [12] to the non-isentropic Euler equations. Next, we derive the Stokes–Dirac structure for the compressible Euler equations in the vorticity–dilatation formulation in Section 6.3 and use this in Section 6.5 to obtain a port-Hamiltonian formulation of the compressible Euler equations in vorticity–dilatation variables. Finally, in Section 7 we finish with some conclusions.

2. Preliminaries

This preliminary section is devoted to summarize the main concepts and techniques that we use throughout this paper in our analysis.

2.1. Review of Hilbert complexes

In this section we discuss the abstract framework of Hilbert complexes, which is the basis of the exterior calculus in Arnold, Falk and Winther [10] and to which we refer for a detailed presentation. We also refer to Brüning and Lesch [13] for a functional analytic treatment of Hilbert complexes.