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Singular quasilinear elliptic problems on unbounded domains

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ABSTRACT

We prove the existence of a solution between an ordered pair of sub and supersolutions for singular quasilinear elliptic problems on unbounded domains. Further, we use this result to establish the existence of a positive solution to the problem

$$\begin{cases} -\Delta_p u = \lambda K(x) f(u) & \text{in } B_1^c, \\ u = 0 & \text{on } \partial B_1, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where $B_1^c = \{x \in \mathbb{R}^n \mid |x| > 1\}$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < n$, λ is a positive parameter, K belongs to a class of functions which satisfy certain decay assumptions and f belongs to a class of $(p-1)$-subhomogeneous functions which may be singular at the origin, namely $\lim_{s\to 0^+} f(s) = -\infty$. Our methods can be also applied to establish a similar existence result when the domain is entire $\mathbb{R}^n$.

1. Introduction

We consider problems of the form

$$\begin{cases} -\Delta_p u = h(x, u) & \text{in } \Omega^c, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, $\Omega \subset \mathbb{R}^n$ is a simply connected bounded domain containing the origin with C^2 boundary $\partial\Omega$, $\Omega^c = \mathbb{R}^n \backslash \overline{\Omega}$ is an exterior domain, and $h : \Omega^c \times (0, \infty) \to \mathbb{R}$ is a Caratheodory function, i.e, $h(x, \cdot)$ is continuous for a.e. $x \in \Omega^c$ and $h(\cdot, s)$ is measurable for all $s \in (0, \infty)$. We prove the existence of a positive weak solution of problem (1) under the assumption of the existence of an ordered pair of sub and supersolutions. We will allow $h(x, s)$ to be singular when $x \in \partial\Omega$ or when $s = 0$. A similar result was established in [1] in case of bounded domains. Our definitions of a weak solution, sub and supersolutions are given below.

By a weak solution of (1), we mean a function $u \in C(\overline{\Omega^c}) \cap C^1(\Omega^c)$ which satisfies

$$\begin{cases} \displaystyle\int_{\Omega^c} |\nabla u|^{p-2}\nabla u \cdot \nabla w = \int_{\Omega^c} h(x, u) w & \text{for all } w \in C_c^\infty(\Omega^c), \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Here, $C_c^\infty(\Omega^c)$ denotes the set of all smooth functions with compact support in Ω^c.

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By a subsolution of (1), we mean a function $\psi \in C(\overline{\Omega^c}) \cap C^1(\Omega^c)$ that satisfies

$$\begin{cases} \int_{\Omega^c} |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla w \le \int_{\Omega^c} h(x,\psi)w & \text{for every } w \in C_c^\infty(\Omega^c),\ w \ge 0 \text{ in } \Omega^c, \\ \psi > 0 & \text{in } \Omega^c, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

By a supersolution of (1), we mean a function $Z \in C(\overline{\Omega^c}) \cap C^1(\Omega^c)$ that satisfies

$$\begin{cases} \int_{\Omega^c} |\nabla Z|^{p-2}\nabla Z \cdot \nabla w \ge \int_{\Omega^c} h(x,Z)w & \text{for every } w \in C_c^\infty(\Omega^c),\ w \ge 0 \text{ in } \Omega^c, \\ Z > 0 & \text{in } \Omega^c, \\ Z = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we prove the following result.

Theorem 1.1. *Let ψ be a subsolution of (1) and Z be a supersolution of (1) such that $\psi \le Z$ in Ω^c and let h satisfy:*

(H_0) *for each bounded set M such that $\overline{M} \subset \Omega^c$, there exists a constant $K_M > 0$ such that for a. e. $x \in M$ and for all s satisfying $\psi(x) \le s \le Z(x)$, $|h(x,s)| \le K_M$.*

Then (1) has a weak solution $u \in C^{1,\beta}_{loc}(\Omega^c)$, for some $\beta \in (0,1)$ such that $\psi \le u \le Z$ in Ω^c.

We note here that the conditions $\psi, Z > 0$ in Ω^c, $\psi = Z = 0$ on $\partial\Omega$ are due to the singularity of h, and can be relaxed to the usual conditions $\psi \le 0, Z \ge 0$ on $\partial\Omega$ in the non singular case. The details of this can be found in the proof of Theorem 1.1. Our methods can be also extended to study the existence of positive weak solutions to problems of the form

$$-\Delta_p u = h(x,u) \quad \text{in } \mathbb{R}^n, \tag{3}$$

where h satisfies (H_0) on $\mathbb{R}^n$. Here, by a weak solution of (3), we mean a function $u \in C^1(\mathbb{R}^n)$ which satisfies a corresponding extension of (2) to $\mathbb{R}^n$. The definitions of sub and supersolutions remain the same except that we do not have to deal with a boundary condition. We obtain:

Theorem 1.2. *Let ψ be a subsolution of (3) and Z be a supersolution of (3) such that $\psi \le Z$ in $\mathbb{R}^n$ and let h satisfy (H_0) on $\mathbb{R}^n$. Then (3) has a weak solution $u \in C^{1,\beta}_{loc}(\mathbb{R}^n)$, for some $\beta \in (0,1)$ such that $\psi \le u \le Z$ in $\mathbb{R}^n$.*

As an application, we consider problems of the form

$$\begin{cases} -\Delta_p u = \lambda K(x) f(u) & \text{in } B_1^c, \\ u = 0 & \text{on } \partial B_1, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{4}$$

where $B_1^c = \{x \in \mathbb{R}^n \mid |x| > 1\}$, $1 < p < n$, λ is a positive parameter, $f : (0,\infty) \to \mathbb{R}$ belongs to a class of continuous functions which are $(p-1)$-subhomogeneous and may be singular at the origin, and K belongs to a class of functions which satisfy certain decay assumptions. Similar problems were studied recently in [2,3], where the authors used Kelvin transformation to reduce (4) to a two point boundary value problem and established the existence of positive radial solutions. In our paper, we study (4) in its original setting in a wider class of possibly non radial functions and establish the existence of a positive weak solution.

Study of positive solutions to boundary value problems of the form

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth domain in $\mathbb{R}^n$, λ is a positive parameter and $f \in C((0,\infty),\mathbb{R})$ has been of great interest over the years. The case when $f(0) < 0$ is referred in the literature as semipositone and it is known to be mathematically challenging to establish the existence of positive solutions. When the domain is bounded, see [4–10] for some existence results in this case. Existence results have been also established in the case when f is singular at the origin, i.e., $\lim_{s\to 0^+} f(s) = -\infty$ (known as an infinite semipositone problem, see [11–13,1,14–17]). When the domain is unbounded there are very few results available in this direction.

We now state our assumptions on f and K.

(H_1) $\lim_{s\to\infty} f(s) = \infty$.

(H_2) $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$ ($(p-1)$-subhomogeneity of f).

(H_3) There exist $b > 0$, $\beta \in (0,1)$ such that $f(s) \ge \frac{-b}{s^\beta}$ $\forall\, s \ge 0$.

(H_4) There exist $\epsilon > 0$, $A > 0$ such that $f(s) \le A$ $\forall s \in (0,\epsilon)$.

(H_5) There exist $C_0 > 0$, $\alpha > n + \beta(\frac{n-p}{p-1})$ such that $0 < K(x) < \frac{C_0}{|x|^\alpha}$ $\forall x \in B_1^c$.

(H_6) $\inf_{|x|=r} K(x) > 0\ \forall r \ge 1$.