Nonlinear Analysis 109 (2014) 236–244

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Combined effects of singular and sublinear nonlinearities in some elliptic problems

Nikolaos S. Papageorgiou [a], Vicenţiu D. Rădulescu [b,c,*]

[a] National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece

[b] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia

[c] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

ARTICLE INFO

Article history:
Received 31 March 2014
Accepted 4 June 2014
Communicated by Enzo Mitidieri

MSC:
35J20
35J60
35J75
58J55

Keywords:
Singular equation
Upper–lower solutions
Truncations

ABSTRACT

We consider a parametric singular Dirichlet equation, with the singular term $u^{-\gamma}$ appearing in the left-hand side. We establish the existence and nonexistence of positive solutions as the parameter $\lambda > 0$ and the exponent $\gamma > 0$ of the singularity vary. In particular, we show that for all $\lambda > 0$ and all $\gamma \geqslant 1$, the problem has no positive solution. Our approach combines truncation arguments with the method of upper and lower solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following parametric singular Dirichlet elliptic problem

$$\begin{cases} -\Delta u + \dfrac{1}{u^{\gamma}} = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega. \end{cases} \tag{P$_\lambda$}$$

Here $\gamma > 0,\ \lambda > 0$ and $f(x,u)$ is a Carathéodory function (that is, for all $u \in \mathbb{R}$ the mapping $x \longmapsto f(x,u)$ is measurable and for a.a. $x \in \Omega$, $u \longmapsto f(x,u)$ is continuous).

The aim of this work is to examine the existence and nonexistence of positive solutions as $\lambda > 0$ and $\gamma > 0$ vary. By a solution of problem (P_λ) we understand the following.

* Corresponding author at: Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania. Tel.: +40 251412615; fax: +40 251411688.

E-mail addresses: npapg@math.ntua.gr (N.S. Papageorgiou), vicentiu.radulescu@imar.ro, vicentiu.radulescu@math.cnrs.fr (V.D. Rădulescu).

http://dx.doi.org/10.1016/j.na.2014.06.016
0362-546X/

Definition 1. A function $u(\cdot)$ is a solution of problem (P_λ) if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $u(x) > 0$ for a.a. $x \in \Omega$, $u^{-\gamma} \in L^1(\Omega)$, $u \geqslant \hat{c}d$ for some $\hat{c} > 0$ with $d(x) = d(x, \partial\Omega)$ and

$$\int_\Omega (Du, Dh)_{\mathbb{R}^N} dx + \int_\Omega \frac{h}{u^\gamma} dx = \lambda \int_\Omega f(x, u) h dx \quad \text{for all } h \in H_0^1(\Omega).$$

This problem with a reaction (right-hand side) independent of u, was investigated by Diaz, Morel and Oswald [1]. Problem (P_λ) differs from the usual singular equations encountered in the literature, where the singular term $u^{-\gamma}$ appears in the right-hand side. So, the problem under consideration in these cases is the following:

$$\begin{cases} -\Delta u = \dfrac{1}{u^\gamma} + \lambda f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega. \end{cases}$$

In fact, the perturbation term $f(x, u)$ has specific form, namely $f(x, u) = f(u) = u^{q-1}$ with $2 < q < 2^*$. For such problems it is proved that the equation exhibits bifurcation phenomena. Namely, there exists a critical parameter value $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ the problem has at least two positive solutions, for $\lambda = \lambda^*$ there is one positive solution and for $\lambda > \lambda^*$ there is no positive solution. We refer to Carl and Perera [2], Ghergu and Rădulescu [3, Chapter 7], Perera and Silva [4,5], and the references therein. In our problem (P_λ), the singular term appears in the reaction with a negative sign and this changes the geometry of the problem.

We introduce the following conditions on the reaction $f(x, u)$:

H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x, 0) = 0$ for a.a. $x \in \Omega$ and

(i) for every $\rho > 0$, there exists a function $a_\rho \in L^\infty(\Omega)_+$ such that
$$f(x, u) \leqslant a_\rho(x) \quad \text{for a.a. } x \in \Omega, \text{ all } 0 \leqslant u \leqslant \rho;$$
(ii) there exists $q \in (1, 2)$ and $c_1 > 0$ such that
$$f(x, u) \geqslant c_1 u^{q-1} \quad \text{for a.a. } x \in \Omega, \text{ all } u \geqslant 0;$$
(iii) $\limsup_{u\to+\infty} \frac{f(x,u)}{u^{q-1}} \leqslant \beta < +\infty$ uniformly for a.a. $x \in \Omega$;
(iv) for a.a. $x \in \Omega$, all $u \geqslant 0$ and all $t \geqslant 1$, we have
$$f(x, tu) \leqslant t f(x, u).$$

In this setting we show that positive solutions exist only for large values of the positive parameter λ. More precisely, we prove the following existence theorem.

Theorem A. *If hypotheses* H *hold and* $\gamma \in (0, 1)$*, then there exists* $\lambda_* > 0$ *such that for all* $\lambda > \lambda_*$ *problem* (P_λ) *admits a solution* $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *with* ${u_\lambda}^{-\gamma} \in L^1(\Omega)$ *and* $u_\lambda \geqslant \hat{c}d$ *for some* $\hat{c} > 0$*; moreover for* $\lambda \in (0, \lambda_*)$ *there is no positive solution.*

Moreover, we investigate also the case $\gamma \geqslant 1$ and prove the following nonexistence result.

Theorem B. *Assume that hypotheses* H *hold,* $\lambda > 0$ *and* $\gamma \geqslant 1$*. Then problem* (P_λ) *has no solution* $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$*.*

Our approach uses the method of upper and lower solutions. For this reason we define what we mean by upper and lower solutions for problem (P_λ).

Definition 2. (a) A function $\bar{u}(\cdot)$ is an upper solution for problem (P_λ), if $\bar{u} \in H_0^1(\Omega)$, $\bar{u}(x) > 0$ for a.a. $x \in \Omega$ and

$$\int_\Omega (D\bar{u}, Dh)_{\mathbb{R}^N} dx + \int_\Omega \frac{h}{\bar{u}^\gamma} dx \geqslant \lambda \int_\Omega f(x, \bar{u}) h dx \quad \text{for all } h \in H_0^1(\Omega),\ h \geqslant 0.$$

(b) A function $\underline{u}(\cdot)$ is a lower solution for problem (P_λ), if $\underline{u} \in H_0^1(\Omega)$, $\underline{u}(x) > 0$ for a.a. $x \in \Omega$ and

$$\int_\Omega (D\underline{u}, Dh)_{\mathbb{R}^N} dx + \int_\Omega \frac{h}{\underline{u}^\gamma} dx \leqslant \lambda \int_\Omega f(x, \underline{u}) h dx \quad \text{for all } h \in H_0^1(\Omega),\ h \geqslant 0.$$

Remark 1. Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we may assume that $f(x, u) = 0$ for a.a. $x \in \Omega$ and for all $u \leqslant 0$. Hypothesis H(iv) is equivalent to saying that for a.a. $x \in \Omega$ the function $u \longmapsto \frac{f(x,u)}{u}$ is nonincreasing. So, the reaction $f(x, \cdot)$ is sublinear.

In addition to the Sobolev space $H_0^1(\Omega)$, we will also use the Banach space

$$C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}.$$

This is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\bar{\Omega}) : u(x) \geqslant 0 \text{ in } \Omega\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(x) > 0 \text{ for all } x \in \Omega,\ \frac{\partial u}{\partial n}(x) < 0 \text{ for all } x \in \partial\Omega \right\},$$

where $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.