Nonlinear Analysis 109 (2014) 245–251

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Blow up in a nonlinear viscoelastic wave equation with strong damping[☆]

Haitao Song[a,b,*], Desheng Xue[a]

[a] Key Laboratory for Magnetism and Magnetic Materials of MOE, Lanzhou University, Lanzhou, Gansu 730000, PR China

[b] School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, PR China

ARTICLE INFO

Article history:
Received 22 October 2013
Accepted 24 June 2014
Communicated by Enzo Mitidieri

Keywords:
Blow-up
Finite time
Hyperbolic
Strong damping
Viscoelastic

ABSTRACT

In this paper, we consider the nonlinear viscoelastic equation:

$$u_{tt} - \triangle u + \int_0^t g(t-\tau) \triangle u(\tau) d\tau - \triangle u_t = \mid u \mid^{p-2} u, \quad \text{in } \Omega \times [0, T],$$

with initial conditions and Dirichlet boundary conditions. For nonincreasing positive functions g, we show the finite time blow up of some solutions whose initial data have arbitrarily high initial energy.

1. Introduction

In [1], Messaoudi considered the following initial–boundary value problem:

$$\begin{cases} u_{tt} - \triangle u + \displaystyle\int_0^t g(t-\tau)\triangle u(\tau)d\tau + au_t|u_t|^{m-2} = bu|u|^{p-2}, & \text{in } \Omega \times (0, \infty) \\ u(x,t) = 0, \quad x \in \partial\Omega, \ t \geq 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, $p > 2$, $m \geq 1$, $a, b > 0$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive nonincreasing function. He proved a blow up result for solution with negative initial energy if $p > m$, and a global result for $p \leq m$. This result was later improved by Messaoudi [2], to certain solutions with positive initial energy. For the problem (1.1) in $\mathbb{R}^n$ and with $m = 2$, Kafini and Messaoudi [3] showed that, under suitable conditions on g and initial data, solution with negative energy blow up in finite time.

In [4], Berrimi and Messaoudi considered

$$u_{tt} - \triangle u + \int_0^t g(t-\tau)\triangle u(\tau)d\tau = u|u|^{p-2}, \quad \text{in } \Omega \times (0, \infty) \tag{1.2}$$

in a bounded domain and $p > 2$. They established a local existence result and showed that the local solution is global and decays uniformly if the initial data are small enough. In [5], Aassila established an asymptotic stability and decay rates, for

[☆] This work was partially supported by the Fundamental Research Funds for the Central Universities Grant (lzujbky-2010-68), and by China Postdoctoral Science Foundation funded project Grant (20100470882), and by Special Financial Grant from China Postdoctoral Science Foundation Grant (201104343), 2w3qe and by the Fundamental Research Funds for the Central Universities Grant (lzujbky-2013-10).

* Corresponding author at: Key Laboratory for Magnetism and Magnetic Materials of MOE, Lanzhou University, Lanzhou, Gansu 730000, PR China. Tel.: +86 13008731107; fax: +86 09318912481.

E-mail address: songhaitao@lzu.edu.cn (H. Song).

http://dx.doi.org/10.1016/j.na.2014.06.012
0362-546X/

solutions of the wave equation in star-shaped domains, were established by combination of memory effect and damping mechanism. In [6], an existence and decay result for viscoelastic problems with nonlinear boundary damping has been proved. For further work on existence and decay of solutions of a viscoelastic equation, we refer to [7–12].

In the absence of the viscoelastic term ($g = 0$), the problem (1.1) has been extensively studied and many results concerning existence and nonexistence have been established. In the further absence of damping mechanism $u_t|u_t|^{m-2}$, any solution with negative initial energy blow up in finite time (see [13,14]). In contrast, in the absence of the source $u|u|^{p-2}$, the damping term $u_t|u_t|^{m-2}$ assures global existence for arbitrary initial data (see [15,16]). The case of linear damping ($m = 2$) and nonlinear source has been first considered by Levine [17,18], who used 'concavity arguments'. He showed that solutions with negative initial energy blow up in finite time. Further more, the interaction between the nonlinear damping and the source terms was studied by Georgiev and Todorova [19], for a bounded domain with Dirichlet boundary conditions. For more related works, we refer the reader to [20–27].

Recently, Song and Zhong [28] study a nonlinear viscoelastic equation with strong damping. They proved a blow-up result for solutions with positive initial energy using the "potential well" theory introduced by Payne and Sattinger [24]. Liang and Gao [29] consider exponential energy decay and blow-up of solutions for a system of nonlinear viscoelastic equations. They proved that for certain initial data in the stable set, the decay rate of the solution energy is exponential. Conversely, for certain initial data in the unstable set, there are solutions with positive initial energy that blow up in finite time. Xu, Yang and Liu [30] study the nonlinear viscoelastic wave equation with strong damping term and dispersive term, and established a global existence and finite time blow-up result.

However until recently there has been very little work on the finite time blow-up of the solution with arbitrary high initial energy for viscoelastic wave equation with strong damping or nonlinear damping. In this work, we are concerned with the following viscoelastic equation with strong damping:

$$\begin{cases} u_{tt} - \triangle u + \int_0^t g(t-\tau) \triangle u(\tau) d\tau - \triangle u_t = |u|^{p-2}u, & \text{in } \Omega \times [0, T], \\ u(x,t) = 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$, is a bounded domain with a smooth boundary $\partial\Omega$. We will show, under suitable conditions on g, that there are solutions of (1.3) with arbitrarily high initial energy that blow up in finite time.

2. Blow up result

We first state a local existence theorem which can be established by the *Faedo–Galerkin* methods. The interested readers are referred to Cavalcanti, Domingos Cavalcanti and Soriano [31] or Georgiev and Todorova [19] for details:

Theorem 2.1. *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Let g be a C^1 function satisfying*

$$1 - \int_0^\infty g(s)ds = l > 0. \tag{2.4}$$

Let $p > 2$ be such that

$$\begin{cases} 2 < p < \infty, & n = 1, 2, \\ 2 < p \le \dfrac{2n}{n-2}, & n \ge 3. \end{cases} \tag{2.5}$$

Then problem (1.3) has a unique local solution

$$u \in C([0, T_m); H_0^1(\Omega)), \quad u_t \in C([0, T_m); L^2(\Omega)) \bigcap L^2([0, T_m], H_0^1(\Omega)),$$

for some $T_m > 0$.

To obtain the blow up result, we need the following lemma [2,28].

Lemma 2.2. *Assume (2.4), (2.5) and (2.7) hold. Let $u(t)$ be a solution of (1.3), then $E(t)$ is nonincreasing, that is $E'(t) \le 0$. Moreover, the following energy inequality holds:*

$$E(t) + \int_s^t \|\nabla u_t(\tau)\|_2^2 d\tau \le E(s), \quad \textit{for } t \ge s \ge 0,$$

where

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u(t)\|_p^p, \tag{2.6}$$

$$(g \circ v) = \int_0^t g(t-\tau)\|v(t) - v(\tau)\|_2^2 d\tau.$$