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Periodic waves of nonlinear Dirac equations[☆]

Yanheng Ding[a], Xiaoying Liu[b,*]

[a] Institute of Mathematics, AMSS, and Hua Loo-Keng Key Lab. Math., Chinese Academy of Sciences, Beijing 100190, PR China

[b] School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, Jiangsu, PR China

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ABSTRACT

Using the variational method, we study existence and multiplicity of periodic solutions of a nonlinear Dirac equation. We establish the variational setting and obtain multiple periodic solutions for the problem with superquadratic and subquadratic nonlinearities, respectively.

1. Introduction and main results

In this paper we devote our attention to the existence and multiplicity of period states to the following stationary Dirac equation

$$-i\sum_{k=1}^{3}\alpha_k\partial_k u + a\beta u + V(x)u = G_u(x,u) \tag{1.1}$$

for $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, where $\partial_k=\partial/\partial x_k$, $a>0$ is a constant, $\alpha_1,\alpha_2,\alpha_3$ and β are 4×4 Pauli–Dirac matrices:

$$\beta=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\qquad \alpha_k=\begin{pmatrix} 0 & \sigma_k\\ \sigma_k & 0\end{pmatrix},\qquad k=1,2,3$$

with

$$\sigma_1=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\qquad \sigma_2=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

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* Corresponding author. Tel.: +86 13776775027.
E-mail addresses: dingyh@math.ac.cn (Y. Ding), xuzhliuxy@163.com (X. Liu).

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The equation arises when one seeks for the standing wave solutions of the nonlinear Dirac equation (see [1])

$$-i\hbar\partial_t\psi = ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi - M(x)\psi + F_\psi(x,\psi). \tag{1.2}$$

Assuming that $F(x, e^{i\theta}\psi) = F(x,\psi)$ for all $\theta \in [0, 2\pi]$, a standing wave solution of (1.2) is a solution of the form $\psi(t,x) = e^{\frac{i\mu t}{\hbar}}u(x)$. It is clear that $\psi(t,x)$ solves (1.2) if and only if $u(x)$ solves (1.1) with $a = mc/\hbar$, $V(x) = M(x)/c\hbar + \mu I_4/\hbar$ and $G(x,u) = F(x,u)/c\hbar$.

There are many papers studying the standing wave of Dirac equations, see, [2–14] and their references. We note that in these papers the solutions u are in $L^2(\mathbb{R}^3, \mathbb{C}^4)$.

A periodic solution of (1.1) may be referred as a standing periodic wave of (1.2). Recall that there have been a great deal of work denoted to the study on existence and multiplicity of periodic solutions to Hamiltonian systems and wave equations, such as [15–23] and their references. However, to our best knowledge, it seems there is no investigation on the periodic solutions of nonlinear Dirac equations. This paper seems the first work on the subject by the variational method.

Let us now describe the results of the present paper. For notational convenience, writing $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ and $\alpha\cdot\nabla = \sum_{k=1}^{3}\alpha_k\partial_k$, we rewrite Eq. (1.1) as

$$-i\alpha\cdot\nabla u + a\beta u + V(x)u = G_u(x,u). \tag{1.3}$$

Suppose that $V(x)$ and $G(x,u)$ are periodic with respect to $x\in\mathbb{R}^3$. Precisely $V(x)$ and $G(x,u)$ satisfy

(V) $V \in C(\mathbb{R}^3, [0,\infty))$, and $V(x)$ is 1-periodic in x_k, $k = 1, 2, 3$.
(G_0) $G \in C^1(\mathbb{R}^3\times\mathbb{C}^4, \mathbb{R})$, and $G(x,u)$ is 1-periodic in x_k, $k = 1, 2, 3$.

We are looking for periodic solutions of Eq. (1.3): $u(x+z) = u(x)$ for any $z\in\mathbb{Z}^3$.

Setting $Q = [0,1]\times[0,1]\times[0,1]$, if u is a solution of Eq. (1.3), its energy will be denoted by

$$\Phi(u) = \int_Q\left[\frac{1}{2}(-i\alpha\cdot\nabla u + a\beta u + V(x)u)\cdot u - G(x,u)\right]dx,$$

where (and in the following) by $v\cdot w$ we denote the scalar product in $\mathbb{C}^4$ of v and w.

First, we consider the superquadratic case. We make the following hypotheses:

(G_1) $G_u(x,u) = o(|u|)$ as $u\to 0$ for $x\in Q$.
(G_2) $G(x,u)|u|^{-2}\to\infty$ as $|u|\to\infty$ for $x\in Q$.
(G_3) $G(x,u)\ge 0$ and there are $\sigma > 3$ and $C, R > 0$ such that

$$|G_u(x,u)|^\sigma \le C\hat{G}(x,u)|u|^\sigma \quad \text{for } x\in Q,\ |u|\ge R,$$

where $\hat{G}(x,u) := \frac{1}{2}G_u(x,u)\cdot u - G(x,u)$.

We have the following conclusions:

Theorem 1.1. *Let (V), (G_0) and (G_1)–(G_3) be satisfied. Then Eq. (1.3) has at least one nontrivial periodic solution in $H^1(Q,\mathbb{C}^4)$.*

Theorem 1.2. *Under the conditions of Theorem 1.1, if moreover $G(x,u)$ is even with respect to u, then Eq. (1.3) possesses a sequence $\{u_n\}$ of periodic solutions with the corresponding energy sequence $\Phi(u_n)\to\infty$.*

Next, we discuss the subquadratic case. We use the following assumptions:

(G_4) $G(x,0) = 0$ and $G_u(x,0) = 0$, $\forall x\in Q$.
(G_5) There are $R_0 > 0$ and $\xi\in(1,2)$ such that

$$0 < G_u(x,u)\cdot u \le \xi G(x,u), \quad \forall x\in Q, |u|\ge R_0.$$

(G_6) There are $d_1 > 0$ and $\tau\in(1,\xi]$ such that

$$G(x,u)\ge d_1|u|^\tau, \quad \forall x\in Q, |u|\ge 1.$$

(G_7) There are $d_2 > 0$ and $\theta\in(1,2)$ such that

$$G(x,u)\ge d_2|u|^\theta, \quad \forall x\in Q, |u|\le 1.$$

(G_8) $\limsup_{|u|\to\infty}|u|^{-1}|G_u(x,u)| = 0$ for $x\in Q$.

Theorem 1.3. *Suppose that (V), (G_0) and (G_4)–(G_8) are satisfied. Then Eq. (1.3) has at least one nontrivial periodic solution in $H^1(Q,\mathbb{C}^4)$.*

Theorem 1.4. *Under the conditions of Theorem 1.3, if in addition $G(x,u)$ is even in u then Eq. (1.3) possesses a sequence of nontrivial periodic solutions $\{u_m\}$ satisfying $0 > \Phi(u_m)\to 0$.*