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Quasi-static crack growth in hydraulic fracture

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ABSTRACT

We present a variational model for the quasi-static crack growth in hydraulic fracture in the framework of the energy formulation of rate-independent processes. The cracks are assumed to lie on a prescribed plane and to satisfy a very weak regularity assumption.

1. Introduction

Hydraulic fracture studies the process of crack growth in rocks driven by the injection of high pressure fluids. The main use of hydraulic fracturing is the extraction of natural gas or oil. In these cases, a fluid at high pressure is pumped into a pre-existing fracture through a wellbore, causing the enlargement of the crack.

In the study of hydraulic fracture, all thermal and chemical effects are usually neglected and the fracturing stimulation is performed only by hydraulic forces, not by explosives, thus the inertial effects are negligible. This justifies the use of quasi-static models.

Numerical simulations for this kind of problems have been presented in various papers, coupling the fluid equation, typically Reynolds' equation, and the elasticity system for the rock, see for instance [1–3]. Particular attention has been given to the tip behavior of a fluid driven crack, see [4,5]. Some models, see, e.g., [6–8], are based on a variational approach introduced by Francfort and Marigo [9] for the *quasi-static* growth of brittle fractures.

While the results of [6–8] are based on a phase field approximation of the crack introduced by Ambrosio and Tortorelli [10], the model presented in this paper is instead built on the sharp-interface version originally developed in [9].

We assume that the rock fills the whole space $\mathbb{R}^3$ and has an initial crack, lying on a plane Σ passing through the origin. The rock is modeled as a linearly elastic, impermeable material and we allow the crack to grow only within Σ. The fluid is pumped through the origin into the region between the crack lips. It is assumed to be an incompressible fluid.

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Since our model is quasi-static, at each time t the fluid and the rock are in equilibrium. This implies that the pressure is uniform in the region occupied by the fluid and exerts a force on the rock through the crack lips. We prove also that the fluid occupies the whole region between the crack lips (see Remarks 3.1 and 4.2). In particular, there is no dry region near the crack edge.

We assume that at every time we know the total volume $V(t)$ of the fluid that has been pumped into the crack up to time t. The mathematical problem is to show that given the function $t \mapsto V(t)$, we can determine at each time the shape and size of the crack, as well as the fluid pressure $p(t)$.

In Section 3 we discuss a simplified version of our model, where we suppose that the rock is homogeneous and isotropic. This justifies the assumption that the time dependent cracks are circular (*penny-shaped* cracks, see, e.g., [11–13]). The main result of this section is the existence of a unique *irreversible quasi-static evolution* (see Theorem 3.9) satisfying a global stability condition at each time as well as an energy-dissipation balance, which involves the stored elastic energy, the energy dissipated by the crack, and the power of the pressure forces exerted by the fluid. Moreover, in this simplified setting the solution can be explicitly given as a function of the volume $V(\cdot)$. The uniqueness follows from a careful analysis of the regularity properties satisfied by the solution.

Finally, in Section 4 we discuss a more general model. In this case, the rock is not necessarily homogeneous or isotropic, so we allow the elasticity tensor $\mathbb{C}$ to be a function of the space variable $x \in \mathbb{R}^3$. Because of the lack of homogeneity and isotropy, we do not expect any symmetry for the crack, so we need to define a new class of admissible cracks, which extends the previous one (see Definition 4.1), keeping some regularity properties of the boundary. Also in this case we prove the existence of an *irreversible quasi-static evolution* (see Theorem 4.4) based on a global stability condition and an energy-dissipation balance. The proof relies on a time discretization procedure introduced in [9] and frequently used in the study of rate-independent processes, see [14].

2. Notation and preliminaries

Let us first give some notation and recall some well known results.

Throughout the paper $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure in $\mathbb{R}^3$ and $\mathcal{K}$ denotes the set of all compact sets of $\mathbb{R}^3$.

Given $K_1, K_2 \in \mathcal{K}$, the Hausdorff distance $d_H(K_1, K_2)$ between K_1 and K_2 is defined by

$$d_H(K_1, K_2) := \max\left\{\max_{x\in K_1} d(x, K_2),\ \max_{x\in K_2} d(x, K_1)\right\}.$$

We say that $K_h \to K$ in the Hausdorff metric if $d_H(K_h, K) \to 0$. The following compactness theorem is well known, see, e.g., [15, Blaschke's Selection Theorem].

Theorem 2.1. *Let K_h be a sequence in $\mathcal{K}$. Assume that there exists $H \in \mathcal{K}$ such that $K_h \subseteq H$ for every $h \in \mathbb{N}$. Then there exist a subsequence K_{h_j} and $K \in \mathcal{K}$ such that $K_{h_j} \to K$ in the Hausdorff metric.*

We say that a set function $K : [0, T] \to \mathcal{K}$ is increasing if $K(s) \subseteq K(t)$ for every $0 \le s \le t \le T$. The following two results about increasing set functions can be found for instance in [16].

Theorem 2.2. *Let $H \in \mathcal{K}$ and let $K : [0, T] \to \mathcal{K}$ be an increasing set function such that $K(t) \subseteq H$ for every $t \in [0, T]$. Let $K^- : (0, T] \to \mathcal{K}$ and $K^+ : [0, T) \to \mathcal{K}$ be the functions defined by*

$$K^-(t) := \overline{\bigcup_{s<t} K(s)} \quad \text{for } 0 < t \le T,$$

$$K^+(t) := \bigcap_{s>t} K(s) \quad \text{for } 0 \le t < T.$$

Then

$$K^-(t) \subseteq K(t) \subseteq K^+(t) \quad \text{for } 0 < t < T.$$

Let Θ be the set of points $t \in (0, T)$ such that $K^+(t) = K^-(t)$. Then $[0, T] \setminus \Theta$ is at most countable and $K(t_h) \to K(t)$ in the Hausdorff metric for every $t \in \Theta$ and every sequence t_h in $[0, T]$ converging to t.

Theorem 2.3. *Let K_h be a sequence of increasing set functions from $[0, T]$ in $\mathcal{K}$. Assume that there exists $H \in \mathcal{K}$ such that $K_h(t) \subseteq H$ for every $t \in [0, T]$ and every $h \in \mathbb{N}$. Then there exist a subsequence, still denoted by K_h, and an increasing set function $K : [0, T] \to \mathcal{K}$ such that $K_h(t) \to K(t)$ in the Hausdorff metric for every $t \in [0, T]$.*

For every open set $\Omega \subseteq \mathbb{R}^3$ we define, as in [17], the space

$$\mathrm{W}^1_{2,6}(\Omega; \mathbb{R}^3) := \{u \in L^6(\Omega; \mathbb{R}^3) : \nabla u \in L^2(\Omega; \mathbb{M}_3)\}$$

equipped with the norm

$$\|u\|_{\mathrm{W}^1_{2,6}(\Omega;\mathbb{R}^3)} := \|u\|_{L^6(\Omega;\mathbb{R}^3)} + \|\nabla u\|_{L^2(\Omega;\mathbb{M}_3)}, \tag{2.1}$$