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On the exact controllability of hyperbolic magnetic Schrödinger equations

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ABSTRACT

In this paper, we address the exact controllability problem for the hyperbolic magnetic Schrödinger equation, which plays an important role in the research of electromagnetics. Typical techniques, such as Hamiltonian induced Hilbert spaces and pseudodifferential operators are introduced. By choosing an appropriate multiplier, we proved the observability inequality with sharp constants. In particular, a genuine compactness–uniqueness argument is applied to obtain the minimal time. In the final analysis, a suitable boundary control is constructed by the systematic Hilbert Uniqueness Method introduced by J. L. Lions. Compared with the micro-local discussion in Bardos et al. (1992), we do not require the coefficients belong to C^∞. Actually, C^1 is already sufficient for the vector potential of the hyperbolic electromagnetic equation.

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RÉSUMÉ

Dans cet article, on considère le problème de contrôlabilité exacte pour l'équation magnétique hyperbolique de Schrödinger, qui joue un rôle important dans la recherche de l'électromagnétisme. Les techniques typiques, tels que les espaces de Hilbert induits de l'opérateur hamiltonien et des opérateurs pseudo-différentiels, sont introduites. En choisissant un multiplicateur approprié, on a démonstré l'inégalité d'observabilité avec des constantes fortes. En particulier, l'argument authentique de compacité-unicité est appliqué pour obtenir le temps minimal. Enfin, un contrôle frontière est construit par la méthode systématique, la méthode hilbertienne de l'unicité introduite par J. L. Lions. Par rapport à la discussion dans Bardos et al. (1992), il n'est pas nécessaire que les coefficients appartiennent à C^∞. En fait, C^1 est déjà suffisante pour le potentiel vecteur de l'équation électromagnétique hyperbolique.

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1. Introduction to hyperbolic magnetic Schrödinger equations and exact controllability

As is known, a magnetic field is produced by electric fields varying in time, spinning of the elementary particles, or moving electric charges, etc. For instance, the earth's magnetic field is a consequence of the movement of convection currents in the outer ferromagnetic liquid of the core. Nowadays, with the fast development of modern technology, electromagnetic theory is widely utilized in medical research of organs' biomagnetism, studying the vortex in the superconductor which carries quantized magnetic flux, and predicting geographical cataclysms, such as earthquakes, volcanic eruptions, and geomagnetic reversal.

From the viewpoint of mathematics, the magnetic field $\mathbf{B}$ is a solenoidal vector field whose field line either forms a closed curve or extends to infinity. In contrast, a field line of the electric field $\mathbf{E}$ starts at a positive charge and ends at a negative charge.

Let $\mathbf{A}(x)$ be the vector potential of $\mathbf{B}$, which does not depend on time, that is, $\mathbf{B} = \nabla \times \mathbf{A}$. Evidently, $\nabla \cdot \mathbf{B} = \operatorname{div} \operatorname{rot}\mathbf{A} = 0$. From one of the Maxwell's equations (μ is the magnetic permeability)

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{B}}{\partial t} = 0,$$

we deduce that $\mathbf{E} = -\nabla\phi$, where the scalar ϕ represents the electric potential. Next we choose an appropriate Lagrangian for the charged particle in the electromagnetic field (q is the electric charge of the particle, and $\mathbf{v}$ is its velocity, m is the mass),

$$\mathscr{L} = \frac{m\mathbf{v}^2}{2} - q\phi + q\mathbf{v} \cdot \mathbf{A}.$$

In particular, the canonical momentum is specified by the equation

$$\mathbf{p} = \nabla_{\mathbf{v}}\mathscr{L} = m\mathbf{v} + q\mathbf{A}.$$

Then we define the classical Hamiltonian by Legendre transform,

$$\mathscr{H} \triangleq \mathbf{p} \cdot \mathbf{v} - \mathscr{L} = m\mathbf{v}^2 + q\mathbf{A} \cdot \mathbf{v} - \left(\frac{m\mathbf{v}^2}{2} - q\phi + q\mathbf{v} \cdot \mathbf{A}\right) = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\phi.$$

In quantum mechanics, we replace $\mathbf{p}$ by $-i\hbar\nabla$, ($\hbar$ is the Planck constant)

$$\mathscr{H} = \frac{(i\hbar\nabla + q\mathbf{A})^2}{2m} + q\phi.$$

When we do not consider the influence from the electric field $\mathbf{E}$, then the above Hamiltonian can be simplified as the differential operator $\mathscr{H}_{\mathbf{A}}^2 \triangleq (i\nabla + \mathbf{A})^2 : \mathcal{H} \to \mathcal{H}^*$. $\mathcal{H}$ and $\mathcal{H}^*$ will be explained in Section 2. This Hamiltonian operator phenomenologically describes a number of behaviors discovered in superconductors and quantum electrodynamics (QED). Ginzburg–Landau equations, Schrödinger equations, Dirac equations and the matrix Pauli operator are famous examples in this respect. For more details, please refer to [1–6].

We are interested in addressing the following variational problem in a suitable function space $\mathcal{U}$, which will be explained later,

$$\min_{u\in\mathcal{U}}\left\{\frac{1}{2}\int_{\Omega}\left(|u_t|^2 - |(i\nabla + \mathbf{A}(x))u|^2 - \phi(x)|u|^2\right)dx\right\}.$$

Let the Lagrangian be

$$\mathcal{L}(t, x_1, \ldots, x_n, u, \bar{u}, u_t, \bar{u}_t, \nabla u, \nabla\bar{u}) \triangleq |u_t|^2 - |(i\nabla + \mathbf{A}(x))u|^2 - \phi(x)|u|^2.$$

The Euler–Lagrangian equation for $\mathcal{L}$ is of the form

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{L}}{\partial u_{x_i}}\right) = 0.$$

In fact, simple calculation leads to

$$\frac{\partial \mathcal{L}}{\partial u} = i\mathbf{A} \cdot \nabla\bar{u} - \mathbf{A}^2\bar{u} - \phi(x)\bar{u}, \qquad \frac{\partial \mathcal{L}}{\partial u_t} = \bar{u}_t, \qquad \frac{\partial \mathcal{L}}{\partial u_{x_i}} = -\bar{u}_{x_i} - ia_i\bar{u}.$$

Consequently, one has

$$u_{tt} + \mathscr{H}_{\mathbf{A}}^2 u + \phi(x)u = 0.$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a time-independent vector potential $\mathbf{A}(x)$. In this work, we mainly address the exact controllability of the hyperbolic magnetic Schrödinger equation in the following form (the modeling process for the

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