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Existence of a ground state solution for an elliptic problem with critical growth in an exterior domain

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ABSTRACT

In this work we prove the existence of a radially symmetric ground state solution for an elliptic equation in an exterior of a ball with Neumann boundary condition.

1. Introduction

In this paper we prove the existence of a radially symmetric ground state solution for the following elliptic equation with Neumann boundary condition

$$\begin{cases} -\Delta u + u = |u|^{q-2}u + |u|^{2^*-2} \quad \text{in } \mathbb{R}^N \backslash \overline{B_R(0)}, \\ \dfrac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial B_R(0), \end{cases} \tag{1}$$

where $N \geq 3$, $B_R(0) \subset \mathbb{R}^N$ is a ball of radius R and center in 0, 2^* is the Sobolev critical exponent, that is,

$$2^* = \frac{2N}{N-2}$$

and $2 < q < 2^*$.

Problem (1) is a variant of the following problem

$$\begin{cases} -\Delta u + u = f(u) \quad \text{in } U, \\ u \in H^1(U), \end{cases} \tag{2}$$

where $U \subset \mathbb{R}^N$ is an unbounded domain with smooth boundary, $N \geq 2$ and f is a continuous function satisfying some hypotheses. In [1], Benci and Cerami studied problem (2) assuming that U is an exterior domain, $N \geq 3$ and $f(u) = |u|^{\eta-1}u$, where $1 < \eta < \frac{N+2}{N-2}$. They showed that (2), with Dirichlet boundary condition, does not have a ground state solution, that

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is, a solution of (2) with the least energy. However, Esteban in [2] proved that the same problem with Neumann boundary condition has a ground state solution. The result obtained by Esteban was extended to the p-Laplacian operator by Alves, Carrião and Medeiros in [3].

In [4], Alves studied problem (2) assuming that U is an exterior domain, $N = 2$ and f with exponential critical growth. He proved the existence of a ground state solution for (2).

In [5], Alves, Montenegro and Souto studied problem (2) considering $U = \mathbb{R}^N$, $N \geq 3$ and f with critical growth. They showed that (2) possesses a ground state solution.

Motivated by [4,3], we complement the studies made in [5,2] in the sense that we consider an exterior of a ball and a nonlinear term with critical growth. Some arguments and estimates found in [5,2] do not hold if we consider an exterior domain and a nonlinearity with critical growth.

The main result of this paper is the following.

Theorem 1.1. *Suppose that $N \geq 4$ and $2 < q < 2^*$ or $N = 3$ and $4 < q < 6$. Then, there exists $R_0 > 0$ such that, for all $R \geq R_0$, problem (1) has a radially symmetric ground state solution.*

One of the arguments to prove Theorem 1.1 is a version of the Mountain Pass Theorem which can be found in [6].

We emphasize that the restriction on q in the case $N = 3$, in the statement of Theorem 1.1, is just because we use a result proved in [7] to obtain an estimate involving the Mountain Pass level. In [7], Miyagaki used such restriction to prove the result in question.

According to our studies, it is not common in elliptic problems involving exterior domains the use of the magic number (the best Sobolev constant), introduced by Brezis and Nirenberg in [8], to establish compactness results. It is unknown if the best constant of the embedding $H^1(\mathbb{R}^N\backslash\overline{\Omega}) \hookrightarrow L^{2^*}(\mathbb{R}^N\backslash\overline{\Omega})$, where $\Omega \subset \mathbb{R}^N$ is a smooth bounded set, is larger than the best constant of the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. In our paper, we conclude that it is true if Ω is a ball with a radius large enough.

This work is organized as follows. In Section 2, we introduce some notations, definitions, the variational formulation of problem (1) and some preliminary results. In Section 3, we introduce an auxiliary problem and some results related to it. In Section 4, we prove an estimate involving Mountain Pass levels and also establish a compactness result for problem (1). In Section 5, we demonstrate Theorem 1.1.

2. Notations and preliminary results

In this section we introduce some notations, definitions, the variational formulation of problem (1) and some preliminary results.

A solution of problem (1) is a function $u \in H^1(\mathbb{R}^N\backslash\overline{B_R(0)})$ satisfying

$$\int_{\mathbb{R}^N\backslash\overline{B_R(0)}} (\nabla u \nabla v + uv)dx = \int_{\mathbb{R}^N\backslash\overline{B_R(0)}} |u|^{q-2}uvdx + \int_{\mathbb{R}^N\backslash\overline{B_R(0)}} |u|^{2^*-2}uvdx,$$

for all $v \in H^1(\mathbb{R}^N\backslash\overline{B_R(0)})$.

In the Hilbert space $H^1(\mathbb{R}^N\backslash\overline{B_R(0)})$, we denote its norm by

$$\|u\| = \left[\int_{\mathbb{R}^N\backslash\overline{B_R(0)}} (|\nabla u|^2 + |u|^2)dx\right]^{\frac{1}{2}}, \quad \text{for } u \in H^1(\mathbb{R}^N\backslash\overline{B_R(0)}). \tag{3}$$

We work in the Hilbert space $X \subset H^1(\mathbb{R}^N\backslash\overline{B_R(0)})$ defined by

$$X := H^1_{rad}(\mathbb{R}^N\backslash\overline{B_R(0)}) = \left\{u \in H^1(\mathbb{R}^N\backslash\overline{B_R(0)}) \ : \ u \text{ is radially symmetric}\right\}$$

endowed with the induced norm of $H^1(\mathbb{R}^N\backslash\overline{B_R(0)})$ defined in (3). The functional $I : X \to \mathbb{R}$ is defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{q}\int_{\mathbb{R}^N\backslash\overline{B_R(0)}} |u|^q dx - \frac{1}{2^*}\int_{\mathbb{R}^N\backslash\overline{B_R(0)}} |u|^{2^*} dx, \quad \text{for } u \in X.$$

The functional I is of class C^1 on space X and critical points of I are solutions of problem (1).

The first result is related to the Mountain Pass shape which proof is well-known.

Lemma 2.1. *The functional $I : X \to \mathbb{R}$ verifies the following statements:*

(i) *There exist constants $\beta, \rho > 0$ such that*

$$I(u) \geq \beta, \quad \forall u \in X, \ \|u\| = \rho;$$

(ii) *There exists $e \in X$ such that*

$$\|e\| > \rho \quad \text{and} \quad I(e) < 0.$$