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On the low regularity solutions and wave breaking for an equation modeling shallow water waves of moderate amplitude

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ABSTRACT

In this paper, we consider a nonlinear evolution equation which models the propagation of surface waves of moderate amplitude in shallow water regime. We first prove the existence of the low regularity solutions for the Cauchy problem of this equation. Then we discuss the initial profiles under which the solutions of this equation blow up in finite time.

1. Introduction

Recently, Constantin and Lannes [1] studied the following general class of equations, which models the propagation of surface waves of moderate amplitude in shallow water regime:

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \varepsilon^2 \iota u^2 u_x + \varepsilon^3 \kappa u^3 u_x + \mu(\alpha u_{xxx} + \beta u_{txx}) = \varepsilon\mu(\gamma u u_{xxx} + \delta u_x u_{xx}), \tag{1.1}$$

where $\iota, \kappa \in \mathbb{R}$, and α, γ, δ and $\beta < 0$ are parameters. The function $u(t, x)$ stands for the free surface elevation. ε and μ are two dimensionless parameters, which represent the amplitude and shallowness parameters, respectively. In [1], Eq. (1.1) was derived as the evolution of the surface elevation of moderate amplitude, which approximates the governing Green–Naghdi equations in the so-called Camassa–Holm (CH) scaling: $\mu \ll 1$ and $\varepsilon = O(\sqrt{\mu})$. While in the so-called long wave scaling: $\mu \ll 1$ and $\varepsilon = O(\mu)$, one can obtain the Korteweg–de Vries (KdV) and Benjamin–Bona–Mahony (BBM) equations. However, the KdV and BBM equations do not have solutions, which blow up in finite time as breaking waves: the solution is bounded, but its slope becomes unbounded [2]. Thus, in order to investigate one of the most fundamental aspects of water waves–wave breaking phenomenon, it inspires us to study the asymptotical equations to the governing equations in the CH scaling, which is larger than the long-wave scaling. Obviously, the CH scaling contains the long-wave scaling, for $\mu \ll 1$. Therefore, Eq. (1.1) covers a wider range of water wave profiles.

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Taking $\iota = \kappa = 0$ in Eq. (1.1), we have the following family of equations:

$$u_t + u_x + \frac{3\varepsilon}{2} u u_x + \mu(\alpha u_{xxx} + \beta u_{txx}) = \varepsilon\mu(\gamma u u_{xxx} + \delta u_x u_{xx}). \tag{1.2}$$

For $\beta < 0$, there are only two equations with a bi-Hamiltonian structure among Eq. (1.2). The first one is the Camassa–Holm equation [3], which can be given by a direct scaling argument for $\beta < 0, \alpha \neq \beta, \beta = -2\gamma$ and $\delta = 2\gamma$ [1]. The CH equation was first derived formally by Fokas and Fuchssteiner [4] as a bi-Hamiltonian generalization of the KdV equation. However, the great interest in the CH equation lies in the fact that Camassa and Holm [3] gave a physical derivation by using an asymptotic expansion directly in the Hamiltonian of Euler's equations in the shallow water regime. Moreover, in contrast to the KdV equation, the CH equation has both solitary waves like solitons [3,5,6], which replicates a feature that is characteristic for the waves of great height-waves of largest amplitude that are exact solutions of the governing equations for water waves [7–10], and solutions which blow up in finite time as wave breaking [11–14]. The CH equation could also be derived as a model for the propagation of axially symmetric waves in hyperelastic rods [15]. It has a bi-Hamiltonian structure [4] and is completely integrable [3,16–19]. Using the same scaling argument as the CH equation, we get the other completely integrable bi-Hamiltonian equation: the Degasperis–Procesi (DP) equation [20], for $\beta < 0, \alpha \neq \beta, \beta = -\frac{8}{3}\gamma$ and $\delta = 3\gamma$ [1]. Analogous to the CH equation, the DP equation has a Lax pair and an infinite sequence of conservation laws [21]. It also admits exact peaked solitons [22–25] and blow-up solutions for a large class of initial profiles [26–29]. Although the DP equation is similar to the CH equation in many aspects, the DP equation has some new features, which cannot be found in the CH equation. It not only has peakons $u(t, x) = ce^{-|x-ct|}, c > 0$, but also shock peaked solutions [30] of the form $u(t, x) = -\frac{1}{t+k} sgn(x)e^{-|x|}, k > 0$. In [27], it is shown that the first blow-up can occur only in the form of wave breaking and shock waves possibly appear afterward. On the other hand, the isospectral problem in the Lax pair of the DP equation is a third-order equation in comparison with the second-order of the CH equation.

As in [1], taking $\iota = -\frac{3}{8}, \kappa = \frac{3}{16}, \alpha = \frac{1}{12}, \beta = -\frac{1}{12}, \gamma = -\frac{7}{24}$ and $\delta = -\frac{7}{12}$ for presenting better structural properties, especially for blow-up phenomena, Eq. (1.1) reads

$$u_t + u_x + \frac{3\varepsilon}{2} u u_x - \frac{3\varepsilon^2}{8} u^2 u_x + \frac{3\varepsilon^3}{16} u^3 u_x + \frac{\mu}{12}(u_{xxx} - u_{txx}) = -\frac{7\varepsilon}{24}\mu(u u_{xxx} + 2u_x u_{xx}). \tag{1.3}$$

A lot of literature has been devoted to Eq. (1.1) (or Eq. (1.3)) of shallow water waves of moderate amplitude. For the local well-posedness of the Cauchy problem to Eq. (1.1), it has been first shown in [1] with any initial data $u_0 \in H^s(\mathbb{R}), s > \frac{5}{2}$. Then, by using Kato's semigroup theory [31], Duruk Mutlubaş [32] proved that Eq. (1.1) is locally well-posed for initial data $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$, as well as for the periodic case [33]. Recently, the Littlewood–Paley decomposition and the nonhomogeneous Besov spaces have been used to study the Cauchy problem of Eq. (1.1) in [34] with initial data $u_0 \in B^s_{p,r}(\mathbb{R}), 1 \leq p, r \leq \infty, s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$. On the other hand, the blow-up mechanism and wave-breaking phenomena for some initial profiles of the line case and the periodic case to Eq. (1.3) have been obtained in [1,33], respectively. Due to the higher order nonlinearities in Eq. (1.3), it is not clear how to get the global strong solutions up to now. As discussed in [35,36] for the CH equation, Zhou and Mu [37] consider the continuation of solutions for Eq. (1.3) after wave breaking. Moreover, there are solitary traveling wave solutions for Eq. (1.3) decaying to zero at infinity [38]. Duruk Mutlubaş and Geyer [39] deduce the orbital stability of solitary waves by using the general method proposed in [40]. Very recently, Gasull and Geyer [41] consider the traveling wave solutions of Eq. (1.3) without the assumption that solitary waves tend to zero at infinity.

However, the existence of low regularity solutions with initial data $u_0 \in H^s(\mathbb{R}), 1 < s \leq \frac{3}{2}$ for Eq. (1.3) has not been discussed yet. Motivated by [14,42], one of the goals of the present paper is to address the question of existence of low regularity solutions for the Cauchy problem to Eq. (1.3) in $H^s(\mathbb{R}), 1 < s \leq \frac{3}{2}$. The method we used is first to regularized Eq. (1.3) and then derive a priori estimates for the solutions of the regularized version of Eq. (1.3), which leads to weak convergence of the solutions. The difficulty here is due to the higher order nonlinearities and its involved structure. On the other hand, we give a sufficient condition on the initial data, which is different from the ones in [1,33], to guarantee that the solutions blow up in finite time in the form of wave breaking.

The remainder of the paper is organized as follows. In Section 2, we recall some required inequalities and the local well-posedness results for Eq. (1.3). In Section 3, we first present a priori estimates of the regularized Eq. (1.3), and then give a sufficient condition for the existence of the low regularity solutions in $H^s(\mathbb{R}), 1 < s \leq \frac{3}{2}$, to Eq. (1.3). In Section 4, we prove that the strong solutions blow up in finite time for some initial data.

Notation. Since our discussion about Eq. (1.3) is on the line $\mathbb{R}$, for simplicity, we omit $\mathbb{R}$ in our notations of function spaces in the sequel. For any function $u = u(x, t) : \mathbb{R} \times [0, T) \to \mathbb{R}$ of two variables with $T > 0$, we denote its Fourier transform, L^p-norm and H^s-norm with respect to x by $\widehat{u} = \widehat{u}(\xi, t), \|u\|_{L^p} = \|u(\cdot, t)\|_{L^p}$ and $\|u\|_{H^s} = \|u(\cdot, t)\|_{H^s}$, respectively. Let $[A, B]$ be the commutator of linear operator A and B.

2. Preliminaries

We begin with a few basic inequalities.

Lemma 2.1 *([43]). If $r > 0$, then $H^r \bigcap L^\infty$ is an algebra. Moreover*

$$\|fg\|_{H^r} \leq c(\|f\|_{L^\infty}\|g\|_{H^r} + \|f\|_{H^r}\|g\|_{L^\infty}),$$

where c is a constant depending only on r.