Nonlinear Analysis 107 (2014) 22–33

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

A new penalization tool in scalar and vector optimizations

Marius Apetrii [a], Marius Durea [a,*], Radu Strugariu [b]

[a] Faculty of Mathematics, "Al. I. Cuza" University, Bd. Carol I, nr. 11, 700506 – Iaşi, Romania

[b] Department of Mathematics and Informatics, "Gheorghe Asachi" Technical University of Iaşi, Bd. Carol I, nr. 11, 700506 – Iaşi, Romania

A R T I C L E   I N F O

Article history:
Received 13 January 2014
Accepted 29 April 2014
Communicated by Enzo Mitidieri

MSC:
47J22
49K40
90C31

Keywords:
Penalization
Directional minimal time function
Pareto efficiency
Necessary optimality conditions
Mordukhovich generalized differentiation

A B S T R A C T

In this work we employ a new method to penalize scalar and vector optimization problems by means of a functional known in the literature as the directional minimal time function. We show that this new approach is useful also for the strong Pareto optimality and then we prove, for this kind of solutions, a result concerning necessary optimality conditions expressed in terms of Mordukhovich generalized differentiation.

1. Introduction

The penalization techniques play a pivotal role in the treatment of many optimization problems with constraints. The main idea behind these procedures is to transform a (local) minimum point of a constrained problem into a (local) minimum point of an unconstrained problem by adding to the objective function a penalty term which somehow contains the constraints of the initial problem. Note that there exist two already classical methods to penalize a given (scalar) optimization problem. If one denotes by M the feasible set (the set of constraints) one direct method is the so-called infinite penalization, where the penalization term is given by the indicator function of M: $i_M(x) = 0$ if $x \in M$ and $i_M(x) = \infty$ if $x \notin M$. The other one is the approach of Clarke (see [1, Proposition 2.4.3]) and uses the distance function associated to M. This approach was recently extended to vector optimization in [2]. If in the infinite penalization technique the penalty term is infinite outside the set M, in Clarke's method the penalization function is real-valued.

The main aim of this paper is to introduce and study a new possibility of penalizing an optimization problem by the use of a functional known in the literature as the directional minimal time function. Compared to the mentioned approaches, our penalization term is hybrid, since outside the set M could take values in $\mathbb{R} \cup \{+\infty\}$. We show that the directional minimal time function is appropriate to penalize the main types of problems with geometric constraints: scalar problems and vectorial problems with solid and nonsolid ordering cones.

Of course, every penalization technique generates necessary optimality conditions by acting together with a (generalized) Fermat rule applied to the unconstrained penalized problem, and this turns to be a general principle in optimization. We

* Corresponding author.
E-mail addresses: mapetrii@uaic.ro (M. Apetrii), durea@uaic.ro (M. Durea), rstrugariu@tuiasi.ro (R. Strugariu).

http://dx.doi.org/10.1016/j.na.2014.04.022
0362-546X/

should say that we are going to present here only one possibility of this general principle built upon our penalization method. In fact, we fully agree with the following remark from [3]: "For us a principle is a generator of theorems, a not yet completely precise statement that can be made into a theorem by filling the technical details and making all the definitions and conditions completely precise. The resulting theorems are versions of the principle. Usually, the choice of technical conditions can be made in more than one way, so a principle has more than one version". Therefore, we illustrate in this paper only some possibilities to "fill the technical details" and to "make the conditions completely precise". We are sure that several other ways are possible depending on the objects which form the chosen framework.

We refer the reader to the following works and the references therein in order to have an image of the topics discussed in this paper: [4–13,2,14].

The structure of this paper is as follows. Section 2 contains the results concerning the new penalization term for scalar and vector optimizations. In the vectorial case, we consider both the cases of weak Pareto and Pareto solutions (the so-called strong Pareto solutions), whence the case of possible empty interior of the ordering cone is contained in our discussion. An example illustrating the results is also included.

In Section 3 we employ the Mordukhovich generalized differentiation objects to obtain necessary optimality conditions for the strong Pareto minimality in a vector optimization problem with the directionally Lipschitz objective function. This approach is just one of the possibilities to combine our penalization technique with a generalized Fermat rule in order to arrive at necessary optimality conditions.

The notations we use are basically standard: we work on normed vector spaces and in this setting, $B(x, r)$, $D(x, r)$, and $S(x, r)$ denote the open, the closed ball and the sphere with center x and radius r, respectively.

2. A penalization term

In [12] the authors have introduced and studied a function called the directional minimal time function. Let us recall this concept. Let X be a real normed linear space, let $u \in X \setminus \{0\}$ be an element and let $\emptyset \neq \Omega \subset X$ be a closed set. One defines $T_u(\cdot, \Omega) : X \to [0, \infty]$,

$$T_u(x, \Omega) := \inf\{t \geq 0 \mid x + tu \in \Omega\}, \tag{2.1}$$

with the convention $\inf \emptyset = +\infty$. Note that this function is a particular case of the minimal time function extensively studied in the literature (see, for instance, [15,16] and the references therein).

The main aim of this paper is to show that the above functional could successfully serve as a penalization term for some general optimization problems with geometric constraint. Note that a particular case of this function was recently used in [17] in order to penalize an optimization problem with generalized functional constraints.

Therefore, we introduce now the constrained scalar optimization problem we are interested in. Let X be a normed vector space and $\emptyset \neq M \subset X$ be a closed set. Suppose that $s : X \to \mathbb{R} \cup \{+\infty\}$ is the scalar objective function and consider the problem

$$(\mathcal{P}_s) \quad \min s(x) \quad \text{such that } x \in M.$$

Denote by $\operatorname{dom} s := \{x \in X \mid s(x) \neq +\infty\}$ the domain of s.

The directional minimal time function defined above provides the exact penalty term for $(\mathcal{P}_s)$, as shown in the next result.

Theorem 2.1. *Suppose that $s : X \to \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz around $\overline{x}$ with constant $L > 0$ and $\overline{x} \in M \cap \operatorname{dom} s$ is a local solution of $(\mathcal{P}_s)$. Then for every $u \in X \setminus \{0\}$, $\overline{x}$ is a local minimum (without constraints) of the function $s + L\|u\| T_u(\cdot, M)$.*

Proof. Take $u \in X \setminus \{0\}$. First, observe that $x \in M$ if and only if $T_u(x, M) = 0$. We show that

$$d(x, M) \leq \|u\| T_u(x, M) \tag{2.2}$$

for every $x \in X$. If $x \in M$ or $T_u(x, M) = +\infty$, there is nothing to prove. Suppose that $x \notin M$ and $T_u(x, M) \in \mathbb{R}$. Taking into account the definition of directional minimal time function in (2.1), for every $\varepsilon > 0$, there exists $t_\varepsilon > 0$ with $x + t_\varepsilon u \in M$ and $t_\varepsilon < T_u(x, M) + \varepsilon$, whence

$$d(x, M) \leq \|x - x - t_\varepsilon u\| \leq \|u\| (T_u(x, M) + \varepsilon).$$

Letting $\varepsilon \to 0$, we deduce (2.2).

Now take $r > 0$ such that on $B(\overline{x}, r)$ all the local properties we have supposed do hold (i.e., s is Lipschitz relative to its domain, $\overline{x}$ is a minimum). Consider $x \in B(\overline{x}, 3^{-1}r) \cap \operatorname{dom} s$. Then, for every $\delta \in (0, 1)$, there exists $u_\delta \in M$ such that

$$\|x - u_\delta\| \leq (1 + \delta) d(x, M) \leq (1 + \delta) \|x - \overline{x}\|,$$

whence

$$\|u_\delta - \overline{x}\| \leq (2 + \delta) \|x - \overline{x}\| < r.$$

Consequently, the Lipschitz property of s leads to

$$s(x) \geq s(u_\delta) - L\|x - u_\delta\| \geq s(u_\delta) - L(1 + \delta) d(x, M) \geq s(\overline{x}) - L(1 + \delta) \|u\| T_u(x, M).$$