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Multilinear Calderón–Zygmund operators with kernels of Dini's type and applications

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ABSTRACT

The main purpose of this paper is to establish a number of results concerning boundedness of multi-linear Calderón–Zygmund operators with kernels of mild regularity. Let T be a multilinear Calderón–Zygmund operator of type $\omega(t)$ with ω being nondecreasing and $\omega \in \text{Dini}(1)$, but without assuming ω to be concave. We obtain the end-point weak-type estimates for multilinear operator T. The multiple-weighted norm inequalities for multilinear operator T and multilinear commutators of T with *BMO* functions are also established.

As applications, multiple-weighted norm estimates for para-products and bilinear pseudo-differential operators with mild regularity and their commutators are obtained.

Moreover, some boundedness properties of the multilinear operators are also established on variable exponent Lebesgue spaces.

Our results improve most of the earlier ones in the literature by removing the assumption of concavity of $\omega(t)$ and weakening the assumption of $\omega \in \text{Dini}(1/2)$ to $\omega \in \text{Dini}(1)$.

1. Introduction and main results

The multilinear Calderón–Zygmund theory was first studied by Coifman and Meyer in [1–3]. This theory was then further investigated by many authors in the last few decades, see for example [4–9], for the theory of multilinear Calderón–Zygmund operators with kernels satisfying the standard estimates. Recently, there are a number of studies concerning multilinear singular integrals which possess rough associated kernels so that they do not belong to the standard Calderón–Zygmund classes. See, for example [10–14] and the references therein. We also mention that the L^p estimates for multi-linear and multi-parameter Coifman–Meyer Fourier multipliers have been established in [15–18].

Recently, Lerner et al. [7] developed a multiple-weight theory that adapts to the multilinear Calderón–Zygmund operators. They established the multiple-weighted norm inequalities for the multilinear Calderón–Zygmund operators and their commutators.

In 2009, Maldonado and Naibo [13] established the weighted norm inequalities, with the Muckenhoupt weights, for the bilinear Calderón–Zygmund operators of type $\omega(t)$, and applied them to the study of para-products and bilinear pseudo-differential operators with mild regularity.

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Motivated by [5,7,13], we will consider the m-linear Calderón–Zygmund operators of type $\omega(t)$ and their commutators, and give some applications to the para-products and the bilinear pseudo-differential operators with mild regularity. In addition, some boundedness properties of the multilinear operators involved on variable exponent Lebesgue spaces are also obtained.

We now give the definition of the multilinear Calderón–Zygmund operators of type $\omega(t)$.

Throughout this paper, we always assume that $\omega(t) : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $0 < \omega(1) < \infty$. For $a > 0$, we say that $\omega \in \mathrm{Dini}(a)$, if

$$|\omega|_{\mathrm{Dini}(a)} := \int_0^1 \frac{\omega^a(t)}{t} dt < \infty.$$

Obviously, $\mathrm{Dini}(a_1) \subset \mathrm{Dini}(a_2)$ provided $0 < a_1 < a_2$, and, if $\omega \in \mathrm{Dini}(1)$ then

$$\sum_{j=0}^{\infty} \omega(2^{-j}) \approx \int_0^1 \frac{\omega(t)}{t} dt < \infty,$$

here and in what follows, $X \approx Y$ means there is a constant $C > 0$ such that $C^{-1}Y \le X \le CY$.

Definition 1.1. A locally integrable function $K(x, y_1, \dots, y_m)$, defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, is called an m-linear Calderón–Zygmund kernel of type $\omega(t)$, if there exists a constant $A > 0$ such that

$$|K(x, y_1, \dots, y_m)| \le \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}} \tag{1.1}$$

for all $(x, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \ne y_j$ for some $j \in \{1, 2, \dots, m\}$, and

$$|K(x, y_1, \dots, y_m) - K(x', y_1, \dots, y_m)| \le \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}} \omega\left(\frac{|x - x'|}{|x - y_1| + \cdots + |x - y_m|}\right) \tag{1.2}$$

whenever $|x - x'| \le \frac{1}{2} \max_{1 \le j \le m} |x - y_j|$, and

$$\begin{aligned} &|K(x, y_1, \dots, y_j, \dots, y_m) - K(x, y_1, \dots, y_j', \dots, y_m)| \\ &\quad \le \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}} \omega\left(\frac{|y_j - y_j'|}{|x - y_1| + \cdots + |x - y_m|}\right) \end{aligned} \tag{1.3}$$

whenever $|y_j - y_j'| \le \frac{1}{2} \max_{1 \le i \le m} |x - y_i|$.

We say $T : \mathscr{S}(\mathbb{R}^n) \times \cdots \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is an m-linear operator with an m-linear Calderón–Zygmund kernel of type $\omega(t)$, $K(x, y_1, \dots, y_m)$, if

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m$$

whenever $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$ and each $f_j \in C_c^\infty(\mathbb{R}^n)$, $j = 1, \dots, m$.

If T can be extended to a bounded multilinear operator from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for some $1 < q, q_1, \dots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1/q$, or, from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ for some $1 < q_1, \dots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1$, then T is called an m-linear Calderón–Zygmund operator of type $\omega(t)$, abbreviated to m-linear ω-CZO.

Obviously, when $\omega(t) = t^\varepsilon$ for some $\varepsilon > 0$, the m-linear ω-CZO is exactly the multilinear Calderón–Zygmund operator studied by Grafakos and Torres in [5]. The linear Calderón–Zygmund operator of type $\omega(t)$ was studied by Yabuta [19]. The bilinear case in this form was considered by Maldonado and Naibo in [13].

In what follows, the letter C always stands for a constant independent of the main parameter and not necessarily the same at each occurrence. A cube Q in $\mathbb{R}^n$ always means a cube whose sides are parallel to the coordinate axes and denote its side length by $\ell(Q)$. For some $t > 0$, the notation tQ stands for the cube with the same center as Q and with side length $\ell(tQ) = t\ell(Q)$. For $1 \le p \le \infty$, let p' be the conjugate index of p, that is, $1/p + 1/p' = 1$. And we will occasionally use the notations $\vec{f} = (f_1, \dots, f_m)$, $T(\vec{f}) = T(f_1, \dots, f_m)$, $d\vec{y} = dy_1 \cdots dy_m$ and $(x, \vec{y}) = (x, y_1, \dots, y_m)$ for simplicity. For a set E and a positive integer l, we will use the notation $(E)^l = \underbrace{E \times \cdots \times E}_{l}$ sometimes.

1.1. Boundedness of m-linear ω-CZO

Our first result on multilinear operators with multilinear Calderón–Zygmund kernel of type ω is the following end-point weak-type estimates on the product of Lebesgue spaces.