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Stability of entire solutions to supercritical elliptic problems involving advection

Craig Cowan *

Department of Mathematical Sciences, University of Alabama in Huntsville, 258A Shelby Center, Huntsville, AL 35899, United States

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ABSTRACT

We examine the equation given by

$$-\Delta u + a(x) \cdot \nabla u = u^p \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $p > 1$ and $a(x)$ is a smooth vector field satisfying some decay conditions. We show that for $p < p_c$, the Joseph–Lundgren exponent, there is no positive stable solution of (1) provided one imposes a smallness condition on a along with a divergence free condition. In the other direction we show that for $N \geq 4$ and $p > \frac{N-1}{N-3}$ there exists a positive solution of (1) provided a satisfies a smallness condition. For $p > p_c$ we show the existence of a positive stable solution of (1) provided a is divergence free and satisfies a smallness condition.

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1. Introduction and results

In this article we are interested in the existence versus nonexistence of positive stable solutions of

$$-\Delta u + a(x) \cdot \nabla u = u^p \quad \text{in } \mathbb{R}^N, \tag{2}$$

where $p > 1$ and $a(x)$ is a smooth vector field satisfying some decay conditions. We now define the notion of stability and for this we prefer to work on a general domain.

Definition 1. Let u denote a nonnegative smooth solution of (2) in an open set $\Omega \subset \mathbb{R}^N$. We say u is a stable solution of (2) in Ω provided there is some smooth positive function E such that

$$-\Delta E + a(x) \cdot \nabla E \geq p u^{p-1} E \quad \text{in } \Omega. \tag{3}$$

We begin by recalling some facts in the case where $a(x) = 0$. There has been much work done on the existence and nonexistence of positive classical solutions of

$$-\Delta u = u^p, \quad \text{in } \mathbb{R}^N. \tag{4}$$

For $N \geq 3$ there exists a critical value of p, given by $p_S = \frac{N+2}{N-2}$, such that for $1 < p < p_S$ there is no positive classical solution of (4) and for $p > p_S$ there exist positive classical solutions, see [1–4]. By definition we call a nonnegative solution u of (4) stable if

$$\int p u^{p-1} \phi^2 \leq \int |\nabla \phi|^2 \quad \forall \phi \in C_c^\infty(\mathbb{R}^N), \tag{5}$$

* Tel.: +1 2568242223.
E-mail address: ctc0013@uah.edu.

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which is nothing more than the stability of u using (3), after using a variational principle. The additional requirement that the solution be stable drastically alters the existence versus nonexistence results. It is known that there is a new critical exponent, the so called Joseph–Lundgren exponent p_c, such that for all $1 < p < p_c$ there is no positive stable solution of (4) and for $p > p_c$ there exist positive stable solutions of (4). The value of the p_c is given by

$$p_c = \begin{cases} \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & N \geq 11 \\ \infty & 3 \leq N \leq 10. \end{cases}$$

The first implicit appearance of p_c was in the work [5] where they examined $-\Delta u = \lambda(u+1)^p$ on the unit ball in $\mathbb{R}^N$ with zero Dirichlet boundary conditions. The exponent p_c first explicitly appeared in the works [6,7] where they examined the stability of radial solutions to a parabolic version of (4). Their results easily imply the existence of a positive radial stable solution of (4) when $p > p_c$ and the nonexistence of positive radial stable solutions in the case of $p < p_c$. More recently there has been interest in finite Morse index solutions of either (4) and the generalized version given by

$$-\Delta u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^N. \tag{6}$$

In [8] they completely classified the finite Morse index solutions of (6) and again the critical exponent p_c was involved. For results regarding singular nonlinearities, general nonlinearities, or quasilinear equation see [9–14].

In the work [15] the nonexistence of nontrivial solutions of

$$-div(\omega_1 \nabla u) = \omega_2 u^p \quad \text{in } \mathbb{R}^N,$$

was examined where ω_i are some nonnegative functions. In the special case where $\omega_1 = \omega_2$ this equation reduces to

$$-\Delta u + \nabla\gamma(x)\cdot\nabla u = u^p \quad \text{in } \mathbb{R}^N, \tag{7}$$

where γ is a scalar function. Even though (7) and (2) are similar a major difference is that (7) is variational in nature; critical points of

$$E(u) = \frac{1}{2}\int e^{-\gamma}|\nabla u|^2 - \frac{1}{p+1}\int e^{-\gamma}|u|^{p+1},$$

are solutions of (7). This variational structure of (7) allows one to prove various nonexistence results for (7) by slightly modifying the nonexistence proofs used in proving similar results for $-\Delta u = u^p$ in $\mathbb{R}^N$. This approach will generally not work for (2) since in general there will not be a variational structure.

In [16] the regularity of the extremal solution, u^*, associated with problems of the form

$$\begin{cases} -\Delta u + a(x)\cdot\nabla u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

was examined for various nonlinearities f. Here $a(x)$ was an arbitrary smooth advection and the main difficulty was to utilize the stability of u^* in a meaningful way. As mentioned earlier, this is not a problem when $a(x)$ is the gradient of a scalar function. The main tool used was the generalized Hardy inequality from [17]. This same approach was extended to more general nonlinearities in [18].

We now list our results.

Theorem 1. *Suppose $3 \leq N \leq 10$ or $N \geq 11$ and $1 < p < p_c$. Suppose $a(x)$ is a smooth divergence free vector field satisfying $|a(x)| \leq \frac{C}{|x|+1}$ with $0 < C$ sufficiently small. Then there is no positive stable solution of (2).*

The next result gives a decay estimate in the case of $p < p_c$. We are including this result since it may allow one to use a Lane–Emden type of change of variables to obtain a nonexistence result without a smallness condition on the advection.

Theorem 2. *Suppose $\frac{N+2}{N-2} < p < p_c$, $a(x)$ is a smooth divergence free vector field with $|a(x)| \leq \frac{C}{|x|+1}$ and $|a| \in L^N(\mathbb{R}^N)$. Then any positive stable solution u of (2) satisfies*

$$\lim_{|x|\to\infty} |x|^{\frac{2}{p-1}} u(x) = 0. \tag{8}$$

The approach to solve Theorem 1 will be to combine the methods used in [8] with the techniques from [16] which relied on generalized Hardy inequalities from [17]. The same approach will be used in the proof of Theorem 2 with an added scaling argument.

Our final result gives an existence result.

Theorem 3. 1. *Suppose $N \geq 4$, $p > \frac{N+1}{N-3}$ and $a(x)$ is some smooth vector field with $|a(x)| \leq \frac{C}{|x|+1}$. If $0 < C$ is sufficiently small there exists a positive solution of (2).*

2. *Suppose $N \geq 11$, $p > p_c$ and let $a(x)$ denote some smooth divergence free vector field with $|a(x)| \leq \frac{C}{|x|+1}$. For $0 < C$ sufficiently small (2) has a positive stable solution.*