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Critical and supercritical higher order parabolic problems in $\mathbb{R}^N$ ☆

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ABSTRACT

Due to the lack of the maximum principle the analysis of higher order parabolic problems in $\mathbb{R}^N$ is still not as complete as the one of the second-order reaction–diffusion equations. While the critical exponents and then a dissipative mechanism in the subcritical case have already been satisfactorily described (see Cholewa and Rodriguez-Bernal (2012)), for problems in the critical or supercritical regime the questions concerning well or ill-posedness, as well as possible dissipative properties of the solutions, have not yet been satisfactorily answered. This article is devoted to the analysis of the higher order parabolic problems in $\mathbb{R}^N$ in the latter case. Focusing on the critical and supercritical regimes we give sufficient "good"-sign conditions proving that the problem is then globally well posed in $L^2(\mathbb{R}^N)$ and even possesses a compact global attractor. On the other hand, for supercritically growing "bad"-signed nonlinearities we show that the problem is ill-posed.

1. Introduction

In this article we consider the Cauchy problem of the form

$$\begin{cases} u_t + \Delta^2 u = f(x,u), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where the nonlinear term is assumed to satisfy a certain critical or supercritical growth condition.

Critical exponents appear naturally when dealing with the well posedness of partial differential equations and they typically arise from Sobolev embeddings. These exponents describe, among others, the largest growth allowed for the nonlinear term in a given class of initial data. As such, critical exponents only account for growth of the nonlinear terms and not for its sign. Thus, they do not distinguish in general between "good" or "bad"-signed nonlinearities, e.g. for nonlinearities

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that for large values of u behave like $\pm|u|^{\rho-1}u$. However the sign of the nonlinear term is known to have a deep impact on the behavior of solutions of nonlinear problems.

For reaction–diffusion equations

$$\begin{cases} u_t - \Delta u = f(x,u), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

and considering for example initial data in the "energy" space $H^1(\mathbb{R}^N)$, if $|f(x,u)| \approx |u|^\rho$ for $|u|$ large, the local existence holds for $\rho \le 1 + \frac{4}{N-2}$, see e.g. [1], while if $f(x,u) \approx |u|^{\rho-1}u$ for $|u|$ large, if $\rho > 1 + \frac{4}{N-2}$ then (1.2) is ill posed, see [2]. Note that (1.2) has naturally associated the energy functional

$$E_{RD}(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} F(x,u),$$

where $F(x,u) = \int_0^u f(x,s)\,ds$ and if $|f(x,u)| \approx |u|^\rho$ for $|u|$ large, then typically $|F(x,u)| \approx |u|^{\rho+1}$ for $|u|$ large. Thus the critical exponent $\rho_c = 1 + \frac{4}{N-2}$ in $H^1(\mathbb{R}^N)$ arises naturally as the largest value of ρ such that $H^1(\mathbb{R}^N) \subset L^{\rho+1}(\mathbb{R}^N)$. For larger ρ the nonlinear term cannot be controlled by the quadratic one.

On the other hand, it is known that when $f(x,u) \approx -|u|^{\rho-1}u$ for $|u|$ large, the Cauchy problem (1.2) is well posed and dissipative for any value of ρ, see [3]. A key point in the analysis of (1.2) in [3] is that, by the maximum principle, for any value of ρ, the solution of (1.2) becomes bounded in $L^\infty(\mathbb{R}^N)$.

Observe that (1.1) has also a natural energy

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 - \int_{\mathbb{R}^N} F(x,u). \tag{1.3}$$

As for (1.2), the critical exponent for (1.1), $\rho_c = 1 + \frac{8}{N-4}$ in $H^2(\mathbb{R}^N)$, arises naturally as the largest value of ρ such that $H^2(\mathbb{R}^N) \subset L^{\rho+1}(\mathbb{R}^N)$. For supercritical nonlinearities, if $\rho > \rho_c$ and if $f(x,u) \approx -|u|^{\rho-1}u$ for $|u|$ large, then (1.3) gives bounds on the solutions in $H^2(\mathbb{R}^N) \cap L^{\rho+1}(\mathbb{R}^N)$. However this is not enough to prove that the solution exists for all times, which depends strongly in proving that the solution remains in $L^\infty(\mathbb{R}^N)$, see [4].

In particular, it was proved in [4, Proposition 3.3] that if $u \in L^\infty((0,T), L^{s_0}(\mathbb{R}^N))$ and

$$s_0 > \frac{N}{4}(\rho - 1) \tag{1.4}$$

then $\|u\|_{L^\infty((\varepsilon,T),L^\infty(\mathbb{R}^N))} \le K(\varepsilon, \|u\|_{L^\infty((0,T),L^{s_0}(\mathbb{R}^N))})$ for any $\varepsilon > 0$ small. But we can take $s_0 = \rho + 1$ on (1.4) only if ρ is subcritical in $H^2(\mathbb{R}^N)$.

For (1.2) the arguments in [3,2] mentioned above use in an essential way the maximum principle, which does not hold for fourth order equations since the kernel of the linear evolution operator changes sign, e.g. [5]. On the other hand, for the second order parabolic problems satisfying some general assumptions, by the Moser–Alikakos technique, [6], suitably weak estimate of the solutions actually implies the L^∞-bound. Neither of these, nor other techniques of getting L^∞-bound on the solutions (see e.g. [7, Theorem II.6.1]), are directly applicable to (1.1), due to the presence of the higher order terms in (1.1).

Hence, one of the motivations for this paper is to extend the results in [3,2] for (1.1). Problems like (1.1) drew lots of attention in the recent years; see e.g. [8–15,4] and references therein. In particular in [15] the local existence of solutions for (1.1) was discussed for several classes of initial data and up to critical nonlinear terms. Also, the global existence and asymptotic behavior were studied in [4] for subcritical "good"-signed nonlinear terms. On the other hand [8–12] paid attention to global solutions of supercritical "bad"-signed nonlinear terms. For this additional restrictions on the size and behavior at infinity of the initial data are required.

In this paper our goals are twofold. For good signed nonlinear terms, we will extend some of the results in [4] to supercritical nonlinearities. For this, we assume that the nonlinear term in (1.1) is of the general form

$$f(x,u) = g(x) + m(x)u + f_0(x,u), \quad x \in \mathbb{R}^N,\ u \in \mathbb{R}, \tag{1.5}$$

where

$$g \in L^2(\mathbb{R}^N) \tag{1.6}$$

and

$$m \in L^r_U(\mathbb{R}^N), \quad r > \frac{N}{4},\ r \ge 2 \tag{1.7}$$

where the uniform space $L^r_U(\mathbb{R}^N)$, for $1 \le r \le \infty$, is defined as

$$L^r_U(\mathbb{R}^N) \overset{\text{def}}{=} \{\phi \in L^r_{\text{loc}}(\mathbb{R}^N) : \|\phi\|_{L^r_U(\mathbb{R}^N)} = \sup_{y\in\mathbb{R}^N} \|\phi\|_{L^r(\{|x-y|\le 1\})} < \infty\},$$

in particular $L^\infty_U(\mathbb{R}^N) := L^\infty(\mathbb{R}^N)$ (see [16]).