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On the regularity of the solutions to the Navier–Stokes equations via the gradient of one velocity component

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ABSTRACT

We improve a regularity criterion for the solutions to the Navier–Stokes equations in the full three-dimensional space involving the gradient of one velocity component. Revising the method used in Pokorný and Zhou (2009, 2010), we show that a weak solution u is regular on $(0, T)$ provided that $\nabla u_3 \in L^t(0, T; L^s)$, where $2/t + 3/s = 19/10$ for $s \in [30/19, 10/3]$ and $2/t + 3/s = 7/4 + 1/(2s)$ for $s \in [10/3, \infty]$. It improves the known results for $s \in [30/19, 150/77)$ and $s \in (10/3, \infty]$.

1. Introduction

We consider the Navier–Stokes equations in the full three-dimensional space, i.e.

$$\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \mathbf{R}^3 \times (0, \infty), \tag{1}$$

$$\nabla \cdot u = 0 \quad \text{in } \mathbf{R}^3 \times (0, \infty), \tag{2}$$

$$u|_{t=0} = u_0, \tag{3}$$

where $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ and $p = p(x, t)$ denote the unknown velocity and pressure and $u_0 = u_0(x) = (u_{01}(x), u_{02}(x), u_{03}(x))$ is a given initial velocity.

It is known that for $u_0 \in L^2_\sigma$ (solenoidal functions from L^2) the problem (1)–(3) possesses at least one global weak solution u satisfying the energy inequality $\|u(t)\|_2^2/2 + \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \le \|u_0\|_2^2/2$ for every $t \ge 0$ (see [1] or [2]). If $u_0 \in W^{1,2}_\sigma$ (solenoidal functions from the standard Sobolev space $W^{1,2}$) then u is known to be regular on some (possibly small) time interval. It is a classical question to ask under which conditions u is regular on an interval $(0, T)$, $T > 0$, i.e. $\nabla u \in L^\infty_{loc}([0, T); L^2)$, $u \in L^2_{loc}(0, T; W^{2,2})$ and (subsequently) $u \in C^\infty_{loc}((0, T) \times R^3)$ (see [2]). There exist many criteria in the literature ensuring the positive answer (see for example [3–11]). In this paper we are interested in criteria involving the gradient of one velocity component ∇u_3. It has not yet been reached in these criteria the level corresponding to the natural scaling of the Navier–Stokes solutions, i.e. $2/t + 3/s = 2$, in contrast, for example, with the situation in criteria involving one direction derivative $\partial_3 u$, where the situation seems to be simpler (see [4,12]). Let us present several recent results: it was proved by Pokorný in [13] that u is regular on $(0, T)$ provided that $\nabla u_3 \in L^t(0, T; L^s)$, $2/t + 3/s \le 3/2$ and

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$s \in [2, \infty]$. In [12] Kukavica and Ziane presented the following criterion: $\nabla u_3 \in L^t(0, T; L^s)$, where $2/t + 3/s \le 11/6$ and $s \in [54/23, 18/5]$. Pokorný and Zhou (see [14,15]) improved the previous results and proved the regularity of u on $(0, T)$ under the condition that $\nabla u_3 \in L^t(0, T; L^s)$ and

$$\frac{2}{t} + \frac{3}{s} \le \frac{19}{12} + \frac{1}{2s}, \quad s \in \left(\frac{30}{19}, \frac{90}{49}\right],$$
$$\frac{2}{t} + \frac{3}{s} \le \frac{53}{18} - \frac{2}{s}, \quad s \in \left(\frac{90}{49}, \frac{54}{29}\right],$$
$$\frac{2}{t} + \frac{3}{s} \le \frac{61}{24} - \frac{5}{4s}, \quad s \in \left(\frac{54}{29}, 2\right),$$
$$\frac{2}{t} + \frac{3}{s} \le \frac{23}{12}, \quad s \in [2, 3),$$
$$\frac{2}{t} + \frac{3}{s} \le \frac{7}{4} + \frac{1}{2s}, \quad s \in \left[3, \frac{10}{3}\right),$$
$$\frac{2}{t} + \frac{3}{s} \le \frac{3}{2} + \frac{4}{3s}, \quad s \in \left[\frac{10}{3}, \infty\right).$$

In this paper we will focus on the method used in [14,15]. The method cleverly combines the estimates of $\partial_3 u$ and $\nabla_h u = (\partial_1 u, \partial_2 u)$. Nevertheless it seems that its potential has not yet been completely fulfilled. By adjusting it we will be able to improve the results from [14,15] for $s \in [30/19, 150/77)$ and $s \in (10/3, \infty]$. Thus, the basic message of this short paper is the following one: it is not excluded that by some optimal application of the method one could get a criterion corresponding to the natural scaling of the Navier–Stokes equations, i.e. $2/t + 3/s = 2$, or at least further improve the results from [14,15] and Theorem 1. Theorem 1 sums up precisely our main result.

Theorem 1. *Let u be a global weak solution to (1)–(3) corresponding to the initial condition $u_0 \in W^{1,2}_\sigma$ and satisfying the energy inequality. Let $T > 0$, $\nabla u_3 \in L^t(0, T; L^s)$ and*

$$\frac{2}{t} + \frac{3}{s} \le \frac{19}{10}, \quad s \in \left[\frac{30}{19}, \frac{10}{3}\right],$$
$$\frac{2}{t} + \frac{3}{s} \le \frac{7}{4} + \frac{1}{2s}, \quad s \in \left(\frac{10}{3}, \infty\right].$$

Then u is regular on $(0, T)$, i.e. $\nabla u \in L^\infty_{loc}([0, T); L^2)$, $u \in L^2_{loc}(0, T; W^{2,2})$ and (subsequently) $u \in C^\infty_{loc}((0, T) \times R^3)$.

Throughout the paper we use several times the following special case of the Troisi inequality (see [16] or [17]): there exists a constant $C > 0$ such that for every $v \in C_0^\infty(\mathbf{R}^3)$

$$\|v\|_6 \le C \prod_{i=1}^{3} \|\partial_i v\|_2^{\frac{1}{3}}. \tag{4}$$

Due to the density argument and the sufficient regularity of u this inequality can be applied to u from Theorem 1. We will also often use a special case of (4): $\|v\|_6 \le C\|\nabla v\|_2$.

In the paper we denote $\int f(x)\, dx$ the integral over the whole three-dimensional space. We write L^p instead of $L^p(\mathbf{R}^3)$. C denotes a generic constant which can change from line to line.

2. Proof of Theorem 1

Proof of Theorem 1. Let $T^* = \sup\{\tau > 0; u \text{ is regular on } (0, \tau)\}$. Since $u_0 \in W^{1,2}_\sigma$, u is regular on some positive time interval and T^* is either equal to infinity (in which case the proof is finished) or it is a positive number and u is regular on $(0, T^*)$. It is sufficient to prove that $T^* \ge T$. We proceed by contradiction and suppose that $T^* < T$. We take $\varepsilon > 0$ sufficiently small (it will be made precise at the end of the proof of Theorem 1) and fix $T_1 \in (0, T^*)$ such that $T^* - T_1 < \varepsilon$ and $\int_{T_1}^{T^*} \|\nabla u(\tau)\|^2 d\tau < \varepsilon$. Taking arbitrarily $T_2 \in (T_1, T^*)$ the proof will be finished if we show that $\|\nabla u(T_2)\|_2 \le C < \infty$, where C is independent of T_2. Actually, the standard extension argument then shows that the regularity of u can be extended beyond T^* and it is contradiction with the definition of T^*. We will use

$$J(T_2)^2 = \sup_{\tau \in (T_1, T_2)} \|\nabla_h(\tau)\|_2^2 + \int_{T_1}^{T_2} \|\nabla \nabla_h u(\tau)\|_2^2 \, d\tau$$

and

$$L(T_2)^2 = \sup_{\tau \in (T_1, T_2)} \|\partial_3 u(\tau)\|_2^2 + \int_{T_1}^{T_2} \|\nabla \partial_3 u(\tau)\|_2^2 \, d\tau.$$