Nonlinear Analysis 104 (2014) 90–108

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Pullback exponential attractors with admissible exponential growth in the past

Radoslaw Czaja [1]

Center for Mathematical Analysis, Geometry, and Dynamical Systems, Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

A R T I C L E I N F O

Article history:
Received 1 December 2013
Accepted 26 March 2014
Communicated by S. Carl

MSC:
primary 37B55
secondary 35L30
35K57

Keywords:
Pullback exponential attractor
Pullback global attractor
Nonautonomous dynamical system
Reaction–diffusion equation
Chafee–Infante equation

A B S T R A C T

For an evolution process we prove the existence of a pullback exponential attractor, a positively invariant family of compact subsets which have a uniformly bounded fractal dimension and pullback attract all bounded subsets at an exponential rate. The construction admits the exponential growth in the past of the sets forming the family and generalizes the known approaches. It also allows to substitute the smoothing property by a weaker requirement without auxiliary spaces. The theory is illustrated with the examples of a nonautonomous Chafee–Infante equation and a time-dependent perturbation of a reaction–diffusion equation improving the results known in the literature.

1. Introduction

In this paper we present a construction of a pullback exponential attractor for an evolution process. A pullback exponential attractor is a positively invariant family of compact subsets which have a uniformly bounded fractal dimension and pullback attract all bounded subsets at an exponential rate. Before the publication of [1] the constructions of a pullback exponential attractor implied that the constructed family is uniformly bounded in the past (see [2–4]). In [1] it was shown that the family may grow sub-exponentially in the past and still have a uniform bound on the fractal dimension. Below we improve the results of that article by allowing the sets to grow even exponentially in the past and still have a uniformly bounded fractal dimension. There are other differences between the results of [1] and ours. First of all, we do not assume the process to be continuous with respect to all the variables, i.e., the function $\{(t,s)\in\mathbb{R}^2\colon t\geq s\}\times V\ni(t,s,u)\mapsto U(t,s)u\in V$ need not be continuous. However, we assume that the operators $U(t,s)$ are Lipschitz continuous within the positively invariant family of bounded absorbing sets $\{B(t)\colon t\in\mathbb{R}\}$. For convenience, we also require that the sets $B(t)$ are closed subsets of the Banach space V. Contrary to assumptions of [1], the sets $B(t)$ may grow exponentially in the past and the absorption of bounded subsets takes place also only in the past (see assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ in Section 2). Moreover, we assume that the process decomposes in the past into a contracting part and a smoothing part (see assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ in Section 2). Under these assumptions we prove the existence of a pullback exponential attractor, which also pullback attracts the family $\{B(t)\colon t\in\mathbb{R}\}$ (see Theorem 2.2). The same assumptions guarantee also the existence of the pullback

E-mail address: czaja@math.ist.utl.pt.

[1] On the leave from: Institute of Mathematics, University of Silesia, Bankowa 14, 40-007 Katowice, Poland.

http://dx.doi.org/10.1016/j.na.2014.03.020

global attractor with uniformly bounded fractal dimension (see Corollary 2.8). From the point of view of applications it seems interesting to substitute the smoothing property $(\mathcal{H}_2)$ by a weaker premise, which does not refer to any auxiliary space. This is done in Corollary 2.6, which together with Theorem 2.2 and Corollary 2.8 constitutes the main results of Section 2.

In Section 3 we consider a nonautonomous Chafee–Infante equation with Neumann boundary conditions

$$\begin{cases} \partial_t u = \triangle u + \lambda u - \beta(t)u^3, \quad t > s,\ x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, \quad t > s,\ x \in \partial\Omega, \\ u(s) = u_s, \quad x \in \Omega, \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^d$. Here we extend the results of [5] by allowing that the real function β tends to 0 in $-\infty$ at an exponential rate. We show that a pullback global attractor and a pullback exponential attractor both exist and have a uniform finite bound on the fractal dimension. However, the diameter of their sections is unbounded in the past and, in the case of a particular β, grows exponentially in the past.

In Section 4 we consider a nonautonomous reaction–diffusion equation with Dirichlet boundary condition

$$\begin{cases} \partial_t u - \triangle u + f(t,u) = g(t), \quad t > s,\ x \in \Omega, \\ u = 0, \quad t > s, x \in \partial\Omega, \\ u(s) = u_s, \quad x \in \Omega, \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^d$ under the assumptions on f considered in [6]. However, here we assume that $g \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ satisfies

$$\|g(t)\|^2_{L^2(\Omega)} \le M_0 e^{\alpha|t|}, \quad t \in \mathbb{R}, \tag{1.1}$$

with $0 \le \alpha < \lambda_1$ and $M_0 > 0$, where $\lambda_1 > 0$ denotes the first eigenvalue of $-\triangle_D$, where $\triangle_D$ is the Laplace operator in $L^2(\Omega)$ with zero Dirichlet boundary condition, while in [6] the function g could have only a polynomial growth. We prove the existence of a pullback exponential attractor and a pullback global attractor in $H^1_0(\Omega)$, both with uniform bound on fractal dimension of their sections, using Corollary 2.6 without the smoothing property.

2. Construction of pullback exponential attractors

Below we present a construction of a family of sets, called a pullback exponential attractor, for an evolution process (see Theorem 2.2), which is a consequence of similar constructions for a discrete semi-process and a discrete process also provided in this section.

We consider an *evolution process* $\{U(t,s): t \ge s\}$ on a Banach space $(V, \|\cdot\|_V)$, i.e., the family of operators $U(t,s): V \to V$, $t \ge s, t, s \in \mathbb{R}$, satisfying the properties

(a) $U(t,s)U(s,r) = U(t,r), t \ge s \ge r$,
(b) $U(t,t) = Id, t \in \mathbb{R}$,

where Id denotes the identity operator on V. If X is a normed space we denote by $\mathcal{O}(X)$ the class of all nonempty bounded subsets of X and by $B^X_R(x)$ the open ball in X centered at x of radius $R > 0$.

Definition 2.1. By a *pullback exponential attractor* for the process $\{U(t,s): t \ge s\}$ on V we call a family $\{\mathcal{M}(t): t \in \mathbb{R}\}$ of nonempty compact subsets of V such that

(i) the family is positively invariant under the process $U(t,s)$, i.e.,

$$U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t), \quad t \ge s,$$

(ii) the fractal dimension in V of the sets forming the family has a uniform bound, i.e., there exists $d \ge 0$ such that

$$\sup_{t\in\mathbb{R}} \dim^V_f(\mathcal{M}(t)) \le d < \infty,$$

(iii) there exists $\omega > 0$ such that every set $D \in \mathcal{O}(V)$ is pullback exponentially attracted at time $t \in \mathbb{R}$ by $\mathcal{M}(t)$ with the rate ω, i.e., for any $D \in \mathcal{O}(V)$ and $t \in \mathbb{R}$ we have

$$\lim_{s\to\infty} e^{\omega s}\mathrm{dist}_V(U(t,t-s)D, \mathcal{M}(t)) = 0, \tag{2.1}$$

where $\mathrm{dist}_V(A,B) = \sup_{x\in A}\inf_{y\in B}\|x-y\|_V$ denotes the Hausdorff semi-distance.