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Uniqueness of radial solutions of semilinear elliptic equations on hyperbolic space

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ABSTRACT

We consider the uniqueness of radial solutions for the semilinear elliptic equation $-\Delta_{\mathbb{H}^d} u + \lambda u - u^{p+1} = 0$ on hyperbolic space $\mathbb{H}^d$. The proof is based on suitable transformations, energy functions and an idea of Yanagida. A similar approach was used by Kwong and Li.
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1. Introduction

We consider the uniqueness of radial solutions for the semilinear elliptic equation on d-dimensional hyperbolic space $\mathbb{H}^d$

$$-\Delta_{\mathbb{H}^d} u + \lambda u - u^{p+1} = 0, \quad x \in \mathbb{H}^d, \tag{1.1}$$

where $\Delta_{\mathbb{H}^d}$ is the Laplace–Beltrami operator on $\mathbb{H}^d$, $\lambda > -\frac{(d-1)^2}{4}$, $0 < p < \frac{4}{d-2}$ and $u : \mathbb{H}^d \to \mathbb{R}$ is a positive radial function with suitable decay at infinity.

Eq. (1.1) is related to solitary waves of nonlinear Schrödinger equation on $\mathbb{H}^d$. Indeed, consider the following NLS

$$iv_t + \Delta_{\mathbb{H}^d} v = |v|^p v, \quad (t, x) \in (0, T) \times \mathbb{H}^d. \tag{1.2}$$

If $v(t, x) = e^{i\lambda t} u(x)$ is a solution of (1.2), then u satisfies (1.1). More results for NLS on $\mathbb{H}^d$, see [1–3]. Concerning the existence of solutions to (1.1), Christianson and Marzuola proved the following.

Theorem 1 (*Theorem 1 and Lemma 6.1 of [4]*). *For $d \geq 2$, $\lambda > -\frac{(d-1)^2}{4}$ and $0 < p < \frac{4}{d-2}$, there exists a positive, radial and decreasing solution $u \in H^1(\mathbb{H}^d) \cap C^2(\mathbb{H}^d)$ satisfying (1.1).*

Mancini and Sandeep in an earlier paper [5] showed the following existence and uniqueness results.

Theorem 2 (*Theorems 1.3 and 1.4 of [5]*). *Let $p > 0$ if $d = 2$ and $0 < p < \frac{4}{d-2}$ if $d \geq 3$. Then (1.1) has a positive entire solution for $\lambda \geq -\frac{(d-1)^2}{4}$. Let $\lambda \geq -\frac{(d-1)^2}{4}$ if $d \geq 3$ and $\lambda > -\frac{2(p+2)}{(p+4)^2}$ if $d = 2$. Then (1.1) has at most one entire positive solution, up to hyperbolic isometries.*

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Here we call a positive solution of the problem (1.1) to be an entire solution if it belongs to the closure of $C_0^\infty(\mathbb{H}^d)$ with respect to the norm (the norm is equivalent to the $H^1(\mathbb{H}^d)$ norm for $\lambda > -\frac{(n-1)^2}{4}$, see [5])

$$\|u\|_\lambda := \left[\int_{\mathbb{H}^d} (|\nabla_{\mathbb{H}^d} u|^2 + \lambda |u|^2)\,\mathrm{d}x\right]^{1/2} \quad \text{for } u \in C_0^\infty(\mathbb{H}^d).$$

Remark 1.1. For the non-existence parts of (1.1), Theorem 1.1 in [5] shows that there do not exist positive solutions in $d \geq 2$ and $\lambda < -\frac{(d-1)^2}{4}$ and there do not exist positive solutions in $H^1(\mathbb{H}^d)$ for $\lambda = -\frac{(d-1)^2}{4}$. [5] also contains partial existence and non-existence results of (1.1) on the critical case, e.g., $p = \frac{2}{d-2}$, please see Section 1 of [5] for more details.

The aim of this paper is to give a different proof for the uniqueness result of [5] in a special case. We have the following result.

Theorem 3. *For $d \geq 3$, $\lambda \geq -\frac{(d-1)^2}{4} + \frac{1}{4}$ and $0 < p < \frac{4}{d-2}$, Eq. (1.1) has a unique positive solution $u \in H^1(\mathbb{H}^d)$, up to hyperbolic isometries.*

Remark 1.2. Since positive solutions are radial (see Theorem 2.1 in [5]), we need only to prove uniqueness for radial solutions.

Remark 1.3. Since $d \geq 3$, the range for λ includes positive part of real line, which is analogous to the classical result of Euclidean case that has been proved by Kwong in [6].

Outline of the proof. First we reduce Eq. (1.1) to an equation on $\mathbb{R}^d$. Since we consider the radial solution, the uniqueness of solutions to (1.1) is equivalent to the problem for an ODE (see, (1.5)), for which we argue by contradiction. Assume that R_1 and R_2 are two distinct positive solutions of (1.5). Then one of the following cases occurs:

- Case 1. R_1 and R_2 intersect infinitely many times.
- Case 2. R_1 and R_2 intersect at most finite times.
  - Subcase 2.1. R_1 and R_2 do not intersect (intersect 0 times).
  - Subcase 2.2. R_1 and R_2 intersect more than once.
  - Subcase 2.3. R_1 and R_2 intersect only once.

Thus, it suffices to rule out the above scenarios. We will rule out these cases by transformations, energy identity and an idea of Yanagida [7], which was used by Kwong and Li in [8] to deal with semilinear problems posed on $\mathbb{R}^d$ with general boundary conditions. The key tool is an exponential decay estimate (see Proposition 1).

1.1. Hyperbolic space and reduction to an Euclidean case

In this subsection we briefly recall the definition of hyperbolic space and the main steps of reduction of (1.1) to an equation on $\mathbb{R}^d$. For more details, please refer to [4].

There are several models of hyperbolic space, we shall use the upper hyperboloid model. Denote

$$\mathbb{H}^d = \{x = (x_0, x_1, \dots, x_d) \in \mathbb{R}^{d+1} \mid [x, x] = 1, x_0 > 0\},$$

where $[x, y] = x_0 y_0 - \cdots - x_n y_n$. Using the polar coordinate,

$$\mathbb{H}^d = \{(\cosh r, \sinh r\omega) \in \mathbb{R}^{d+1} \mid r > 0, \omega \in \mathbb{S}^{d-1}\},$$

one can obtain the following expression of $\Delta_{\mathbb{H}^d}$:

$$\Delta_{\mathbb{H}^d} = \partial_r^2 + (d-1)\frac{\cosh r}{\sinh r}\partial_r + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}. \tag{1.3}$$

The metric on $\mathbb{H}^d$ is given by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \sinh^2 r\,\mathrm{d}\omega^2,$$

and the integration of function f on $\mathbb{H}^d$ is

$$\int_{\mathbb{H}^d} f(x)\,\mathrm{d}x = \int_0^\infty \int_{\mathbb{S}^{d-1}} f(r, \omega) \sinh^{d-1} r\,\mathrm{d}r\,\mathrm{d}\omega.$$

Here $\mathbb{S}^{d-1} = \{\omega \in \mathbb{R}^d \mid |\omega| = 1\}$ is the unit sphere on $\mathbb{R}^d$ and $\mathrm{d}\omega^2$ is the metric on the sphere $\mathbb{S}^{d-1}$.

The Sobolev space $H^1(\mathbb{H}^d)$ is defined by

$$H^1(\mathbb{H}^d) = \{f \in L^2(\mathbb{H}^d) \mid |\nabla_{\mathbb{H}^d} f| \in L^2(\mathbb{H}^d)\}$$

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