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$C^{1,1}$ solution of the Dirichlet problem for degenerate k-Hessian equations

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ABSTRACT

In this paper, we prove the existence of the $C^{1,1}$-solution to the Dirichlet problem for degenerate elliptic k-Hessian equations $S_k[u]=f$ under a condition which is weaker than the condition $f^{1/k}\in C^{1,1}(\bar{\Omega})$.

1. Introduction

In this work, we study the following Dirichlet problem for the k-Hessian equation

$$\begin{cases} S_k[u]=f(x) & \text{in } \Omega, \\ u=\varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $S_k[u]$ is defined as follows:

$$S_k[u]=\sigma_k(\lambda), \quad k=1,\ldots,n,$$

where $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$, λ_i is the eigenvalue of the Hessian matrix (D^2u), and

$$\sigma_k(\lambda)=\sum_{1\le i_1<\cdots<i_k\le n}\lambda_{i_1}\cdots\lambda_{i_k} \tag{1.2}$$

is the k-th elementary polynomial. Note that the case $k=1$ corresponds to the Poisson equation, while for $k=n$, that is the Monge–Ampère equation $\det D^2u=f$.

The nonlinear equation of (1.1) is referred to as non-degenerate when the function f is positive, it is degenerate elliptic if f is non-negative and allowed to vanish somewhere in $\overline{\Omega}$.

The non-degenerate k-Hessian equations were firstly studied by Caffarelli, Nirenberg and Spruck [1]. They proved that if $f\in C^{1,1}(\overline{\Omega})$, $f>0$, $\partial\Omega$ and φ were sufficiently smooth, (1.1) had a unique $C^{3,\alpha}$ k-admissible solution. For the degenerate case, Ivochkina, Trudinger and Wang [2] studied a class of fully nonlinear degenerate elliptic equations which depended only on the eigenvalues of the Hessian matrix. This kind of equations include the k-Hessian equations. They got the à priori

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estimate with the condition $f^{1/k} \in C^{1,1}(\overline{\Omega})$. In particular, their estimation of second order derivatives was independent of $\inf_{\Omega} f$. Thus, the condition $f^{1/k} \in C^{1,1}(\overline{\Omega})$ implied the existence of $C^{1,1}$-solutions to the degenerate k-Hessian equations. Then, the regularity of the degenerate k-Hessian equations paused at $C^{1,1}$. For Monge–Ampère equations, Hong, Huang and Wang [3] gave a special condition to the smooth solution for the 2-dimensional Monge–Ampère equation. We can find that even f is analytic, the solution may be not in C^2, see for example [4]. For degenerate k-Hessian equations, some papers concentrated on the convexity of the solutions [5].

In this work, we want to improve these results of $C^{1,1}$-regularity with a condition weaker than $f^{1/k} \in C^{1,1}(\overline{\Omega})$. To state our results, we set the following condition for the function f which is the right hand side term of the equations.

Condition (H). *Assume that $f \in C^{1,1}(\overline{\Omega})$, $f \geq 0$ and there exists a constant $C_0 > 0$ such that*

$$|Df(x)| \leq C_0 f^{1-\frac{1}{k}}(x) \quad \forall x \in \Omega,$$

and for any vector $\xi \in \mathbb{S}^{n-1}$,

$$f(x) f_{\xi\xi}(x) - \left(1 - \frac{1}{k}\right) f_{\xi}^2(x) \geq -C_0 f^{2-\frac{1}{k}}(x) \quad \forall x \in \overline{\Omega},$$

where $f_{\xi}(x) = \frac{\partial f}{\partial \xi}(x)$, $f_{\xi\xi}(x) = \frac{\partial^2 f}{\partial \xi^2}(x)$.

We will show that Condition (H) is weaker than $f^{\frac{1}{k}} \in C^{1,1}(\overline{\Omega})$ in Section 2. Indeed, for the case of 3-dimension we can give an example that $f \geq 0$ is analytic and $f^{\frac{1}{2}}$ is only Lipschitz continuous, while f satisfies Condition (H).

Our main result is stated as follows.

Theorem 1.1. *Assume that Ω is a bounded $(k-1)$-convex domain in $\mathbb{R}^n$ with $C^{3,1}$ boundary $\partial\Omega$, $f \geq 0$, f satisfies Condition (H), and $\varphi \in C^{3,1}(\partial\Omega)$. Then the Dirichlet problem (1.1) has a unique k-admissible solution $u \in C^{1,1}(\overline{\Omega})$. Moreover,*

$$\|u\|_{C^{1,1}(\overline{\Omega})} \leq C,$$

where C depends only on n, k, Ω, $\|f\|_{C^{1,1}(\overline{\Omega})}$, $\|\varphi\|_{C^{3,1}(\partial\Omega)}$ and C_0. In particular, C is independent of $\inf_{\Omega} f$.

We will recall the notions of $(k-1)$-convexity and k-admissibility in Section 2.

In [6], for the degenerate Monge–Ampère equations, P. Guan introduced a condition weaker than $f^{1/n} \in C^{1,1}(\overline{\Omega})$. So that our Condition (H) is inspired by Guan's Condition (C) in [6].

In Section 2, we will recall some definitions and some known results, then we will give the sketch of the proof to the main theorem. Then, the rest of this paper (Sections 3–5) is to establish the uniform à priori estimates for the approximate solutions.

2. Sketch of the proof to the main theorem

In this section, we firstly recall some definitions and known results. Then we will present the sketch of the proof of Theorem 1.1.

Preliminaries

Firstly, we recall some definitions about the k-Hessian equations.

Definition 2.1 (*[7]*)**.** We say a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is k-admissible if

$$\lambda(D^2 u) \in \overline{\Gamma}_k,$$

where Γ_k is an open symmetric convex cone in $\mathbb{R}^n$, with vertex at the origin, given by

$$\Gamma_k = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \ \sigma_j(\lambda) > 0, \ \forall j = 1, \ldots, k\},$$

where $\sigma_j(\lambda)$ is defined by (1.2).

The geometry condition for $\Omega \subset \mathbb{R}^n$ is (see [1]),

Definition 2.2. We say that Ω is $(k-1)$-convex if there exists a constant $c > 0$, such that, for any $x \in \partial\Omega$

$$\sigma_j(\kappa)(x) \geq c > 0 \quad j = 1, \ldots, k-1,$$

where $\kappa = (\kappa_1, \ldots, \kappa_{n-1})$, $\kappa_i(x)$ is the principal curvature of $\partial\Omega$ at x. When $k = n$, it is the usual convexity.

The weak solution to the k-Hessian equation is defined as follows.

Definition 2.3 (*[8]*)**.** A function $u \in C^0(\Omega)$ is called an admissible weak solution of Eq. (1.1) in the domain Ω, if there exists a sequence $\{u_m\} \subset C^2(\Omega)$ of k-admissible functions such that

$$u_m \to u \quad \text{in } C^0(\Omega), \qquad S_k(u_m) \to f \quad \text{in } L^1_{\text{loc}}(\Omega).$$