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On the exponential type explosion of Navier–Stokes equations

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ABSTRACT

The classical results on the explosion of the maximal solution of incompressible Navier–Stokes equations are of type $c(T^* - t)^{-\sigma_0}$ for some $\sigma_0 > 0$. Inspired by the works Benameur and Selmi (2012) [15], Chemin (2004) [16], we use the Sobolev–Gevrey spaces to get better explosion results, precisely if $e^{a|D|^{1/\sigma}} u^0 \in H^s(\mathbb{R}^3)$, then $|e^{a|D|^{1/\sigma}} u(t)|_{H^s}$ is at least of the order $(T^* - t)^{-\sigma_1} \exp(c(T^* - t)^{-\sigma_2})$ for some $\sigma_1 > 0$ and $\sigma_2 > 0$. Fourier analysis and standard techniques are used.

1. Introduction

The incompressible Navier–Stokes system in Cartesian coordinates is given by:

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u = -\nabla p, & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ u(0) = u^0 & \text{in } \mathbb{R}^3, \end{cases} \tag{NS}$$

where $\nu > 0$ is the viscosity of the fluid, $u = u(t, x) = (u_1, u_2, u_3)$ and $p = p(t, x)$ denote respectively, the unknown velocity and the unknown pressure of the fluid at the point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, $(u \cdot \nabla u) := u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$, and $u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))$ is a given initial velocity. If u^0 is quite regular, the pressure p is determined. The system (NS) has the scaling property: If $u(t, x)$ is a solution of the initial data $u^0(x)$, then for any $\lambda > 0$, $\lambda u(\lambda^2 t, \lambda x)$ is a solution of (NS) with the initial data $\lambda u^0(\lambda x)$. Our problem is the type of the blow-up criterion of the solution if the maximal time T^* is finite. Precisely, the question posed by K. Ammari [1]: Is the type of explosion due to the chosen space or to the non-linear part of the Navier–Stokes equations? In the literature, there are several authors who have studied the problem of explosion of a non-global solution of (NS) system, and all the obtained results do not exceed $C(T^* - t)^{-\sigma}$ for some $\sigma \geq 0$, precisely: if u is a maximal solution of (NS) in $\mathcal{C}([0, T^*), \mathbf{B})$ with $T^* < \infty$, then $C(T^* - t)^{-\sigma} \leq \|u(t)\|_{\mathbf{B}}$. The classical result due to J. Leray [2,3]: if u is a non regular solution of the system (NS) and T^* is the first time where u is not regular,

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then $c\nu^{3/4}(T^*-t)^{-1/4} \le \|u(t)\|_{\dot H^1}$. Also, if $u^0 \in H^s(\mathbb{R}^3)$, with $s > 5/2$, T. Kato [4] showed that there is an existence and uniqueness for a local solution of (NS) in $\mathcal{C}([0,T]; H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0,T]; H^{s-1}(\mathbb{R}^3))$. For more information, J. Beale, T. Kato, A. Majda [5], and T. Kato, G. Ponce [6] proved the following blow-up result: let $u \in \mathcal{C}([0,T^*); H^s(\mathbb{R}^3))$ be a maximal solution of (NS) given in [4] with T^* is finite, then $\|\nabla \times u\|_{L^1([0,T^*),L^\infty(\mathbb{R}^3))} = +\infty$. To prove this result, the authors used the following estimate $\partial_t\|u(t)\|^2_{H^s} \le C\|\nabla u(t)\|_{L^\infty}\|u(t)\|^2_{H^s}$, which gives $c(T^*-t)^{-1} \le \|u(t)\|_{H^s}$. To improve this result, some authors treat the dependence of the growth of $\|u(t)\|_{H^s}$ with respect to the index of regularity. In [7], the author proved that, if $u \in \mathcal{C}([0,T^*); H^k(\mathbb{R}^3))$ with $k > 5/2$ an integer, is the maximal solution of Euler or Navier–Stokes equations, then $C(T^*-t)^{-2k/5} \le \|u(t)\|_{\dot H^k}$. In [8], I showed that the quality of the explosion depends on the index of regularity. Precisely, let $u \in \mathcal{C}([0,T^*), H^s(\mathbb{R}^3))$, $(s > 5/2)$, be a maximal solution of (NS), given by [4]. Suppose that $T^* < \infty$, then $C(T^*-t)^{-s/3} \le \|u(t)\|_{\dot H^s}$, in which we can take $s > 3/2$. In [9], for the periodic case, I proved that, if the maximal solution $u \in \mathcal{C}([0,T^*); H^s(\mathbb{T}^3))$, $s > 5/2$, of Navier–Stokes equations, then $C(T^*-t)^{-2s/5} \le \|u(t)\|_{\dot H^s}$, and other results in $\dot H^{s'}$ for $s' \in [1,s]$. So, the idea is to find a functional space $\mathbf{B}_{\alpha,s}(\mathbb{R}^3)$ which has the form

$$\mathbf{B}_{\alpha,s}(\mathbb{R}^3) = \left\{ f \in L^2(\mathbb{R}^3);\ \int_{\mathbb{R}^3} (1+|\xi|^2)^s \big(\alpha(|\xi|)\big)^2 |\widehat{f}(\xi)|^2 d\xi < \infty \right\},$$

with $s > 3/2$ and the function α is defined and increasing from $[0,\infty)$ to $(0,\infty)$. The choice of $s > 3/2$ is not optimal for the existence and uniqueness, we can take $s > 1/2$. But this condition is necessary to estimate $\|\widehat{u}(t)\|_{L^1}$ for finding a new blow up result. This space is equipped with the norm

$$\|f\|_{\mathbf{B}_{\alpha,s}} = \|\alpha(|D|)f\|_{H^s}$$

and the associated inner product

$$\langle f/g\rangle_{\mathbf{B}_{\alpha,s}} = \langle \alpha(|D|)f/\alpha(|D|)g\rangle_{H^s}.$$

For the existence and uniqueness of the solution, a sufficient condition is the following

$$\alpha(|\xi|) \le C\alpha(|\xi-\eta|)\alpha(|\eta|), \quad \forall \xi,\eta \in \mathbb{R}^3. \tag{1}$$

We can suppose that $C = 1$, simply to change α by $C\alpha$. If we take $\beta = \ln(\alpha)$, the condition (1) becomes

$$\beta(|\xi|) \le \beta(|\xi-\eta|) + \beta(|\eta|), \quad \forall \xi,\eta \in \mathbb{R}^3.$$

A sufficient condition is

$$\beta(|\xi|) \le a|\xi|, \quad \forall \xi \in \mathbb{R}^3$$

where $a > 0$.

Therefore, we consider a function α which satisfies

$$\alpha(|\xi|) \le e^{a|\xi|}, \quad \forall \xi \in \mathbb{R}^3.$$

Then, we choose the following family

$$\alpha(y) = e^{ay^r}, \quad \forall y \in [0,\infty),$$

with $a > 0$ and $r \in (0,1]$. To address the main issue, we will use the Sobolev–Gevrey space $H^s_{a,\sigma}(\mathbb{R}^3)$, $a \ge 0$ and $s \in \mathbb{R}$, $\sigma = \frac{1}{r} \in [1,\infty)$, with the norm

$$\|f\|_{H^s_{a,\sigma}} = \|e^{a|D|^{1/\sigma}} f\|_{H^s}.$$

Such Sobolev–Gevrey spaces have been used in other contexts, e.g., the recent works [10–12].

Now we are ready to state the main result.

Theorem 1.1. *Let a, s, $\sigma \in \mathbb{R}$ such that $a > 0$, $s > 3/2$ and $\sigma > 1$. Let $u^0 \in \left(H^s_{a,\sigma}(\mathbb{R}^3)\right)^3$ such that* $\operatorname{div} u^0 = 0$. *Then, there is a unique time $T^* \in (0,\infty]$ and a unique solution $u \in \mathcal{C}([0,T^*), H^s_{a,\sigma}(\mathbb{R}^3))$ of Navier–Stokes equations (NS) such that $u \notin \mathcal{C}([0,T^*], H^s_{a,\sigma}(\mathbb{R}^3))$. Moreover, if $T^* < \infty$, then*

$$C_1(T^*-t)^{-s/3}\exp\left(aC_2(T^*-t)^{-\frac{1}{3\sigma}}\right) \le \|u(t)\|_{H^s_{a,\sigma}}, \quad \forall t \in [0,T^*), \tag{2}$$

where $C_1 = C_1(s,u^0,\sigma) > 0$ and $C_2 = C_2(s,u^0,\sigma) > 0$.

In this short paragraph, we explain our choice and its novelty. In the literature of the study of Navier–Stokes equations, there are different types of spaces that are used. The first type is the homogeneous functional space, that is to say $\|\lambda f(\lambda\,\cdot)\|_E = \lambda^\mu\|f\|_E$, $\forall \lambda > 0$; For example: $\dot H^s(\mathbb{R}^3)$, $L^p(\mathbb{R}^3)$, and $\dot{\mathcal{C}}^r(\mathbb{R}^3)$. There are also spaces that are almost homogeneous (homogeneous for only $\lambda = 2^k$, $k \in \mathbb{Z}$) and satisfy, for all $\lambda > 0$, $\|\lambda f(\lambda\,\cdot)\|_E \simeq \lambda^\mu\|f\|_E$, that is to say $c_1\lambda^\mu\|f\|_E \le \|\lambda f(\lambda\,\cdot)\|_E \le c_2\lambda^\mu\|f\|_E$. The homogeneous Besov spaces give a typical example for this kind. In the same frame,