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Regularity criteria with angular integrability for the Navier–Stokes equation

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ABSTRACT

We give new *a priori* assumptions on weak solutions of the Navier–Stokes equation so as to be able to conclude that they are smooth. The regularity criteria are given in terms of mixed radial–angular weighted Lebesgue space norms.

1. Introduction and main results

We consider the Cauchy problem on $(0, T) \times \mathbb{R}^n$

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \Delta u = -\nabla P \\ \nabla \cdot u = 0 \\ u(x, 0) = u_0(x). \end{cases} \tag{1.1}$$

It describes the motion of a viscous incompressible fluid in the absence of external forces, where u is the velocity field and P is the pressure.

The first equation is the Newton law while the second follows by the incompressibility of the fluid. In order to require incompressibility at time $t = 0$ it is necessary to restrict to initial data u_0 such that $\nabla \cdot u_0 = 0$.

We shall use the same notation for the norm of scalar, vector or tensor quantities, for instance:

$$\|P\|_{L^2}^2 := \int P^2 \, dx, \qquad \|u\|_{L^2}^2 := \int \sum_{i=1}^n u_i^2 \, dx, \qquad \|\nabla u\|_{L^2}^2 := \int \sum_{i,j=1}^n (\partial_i u_j)^2 \, dx$$

and we often write simply $u \in L^2(\mathbb{R}^n)$ instead of $u \in [L^2(\mathbb{R}^n)]^n$.

The well-posedness of (1.1) is still open even if many partial results have been obtained. In [1,2] the authors proved the global existence of weak solutions for initial data in L^2 but a satisfactory well-posedness theory is basically developed only in the case of small initial data or data with a peculiar geometric structure.

In this scenario it is useful to establish *a priori* conditions under which uniqueness and regularity of the weak solutions are guaranteed. Results of this kind are usually called regularity criteria.

In this paper we focus on some classical regularity criteria [3–6] and their extension to the setting of weighted Lebesgue spaces [7]. In particular we show how the results in [7] can be improved under the hypothesis of additional angular integrability.

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The regularity is basically ensured by boundedness assumptions on quantities like u, ∇u, $\nabla \times u$ in suitable critical spaces. A simple regularity criterion is for instance

$$\|u\|_{L^s_T L^p_x} := \left(\int_0^T \left(\int_{\mathbb{R}^n} |u(t,x)|^p \, dx \right)^{\frac{s}{p}} dt \right)^{\frac{1}{s}} < +\infty, \qquad \frac{2}{s} + \frac{n}{p} \leq 1. \tag{1.2}$$

Notice that in the endpoint case (1.2) is invariant with respect to

$$u(t,x) \to \lambda u(\lambda^2 t, \lambda x), \tag{1.3}$$

that is the natural scaling of (1.1). In [4] smoothness in space variables has been obtained in the case $\frac{2}{s} + \frac{n}{p} < 1$, while the endpoints have been fixed in [8,9,5,6,10]. We recall the following

Definition 1.1 ([3]). We say that a point $(\bar{t}, \bar{x}) \in (0,T) \times \mathbb{R}^3$ is *regular* for a solution $u(t,x)$ of (1.1) if u is essentially bounded on a neighborhood of $(\bar{t}, \bar{x})$. (In this case one can prove that $u(t,x)$ is smooth near $(\bar{t}, \bar{x})$, see for instance [4].) We say that a set is *regular* if all its points are regular.

Let us also recall that $(0,T) \times \mathbb{R}^n$ is regular provided that (1.2) is satisfied with $2/s + n/p = 1$ (see for instance [5,6]).

Then we focus on the weighted norm approach:

Theorem 1.2 ([7]). *Let $n \geq 3$ and $u_0 \in L^2(\mathbb{R}^n)$ be a divergence free vector field. Let then u be a weak solution of (1.1) and $\bar{x} \in \mathbb{R}^n$ such that*

$$\left\| \, |x - \bar{x}|^{1-\frac{n}{2}} u_0 \right\|_{L^2_x} < +\infty, \tag{1.4}$$

$$\| \, |x - \bar{x}|^{\alpha} u(x,t) \|_{L^s_T L^p_x} < +\infty, \tag{1.5}$$

with

$$\begin{aligned} &\frac{2}{s} + \frac{n}{p} = 1 - \alpha, \quad -1 \leq \alpha < 1 \\ &\frac{2}{1-\alpha} < s < +\infty, \qquad \frac{n}{1-\alpha} < p < +\infty; \end{aligned} \tag{1.6}$$

or

$$\| \, |x - \bar{x}|^{\alpha} u(x,t) \|_{L_T^{\frac{2}{1-\alpha}} L^\infty_x} < +\infty, \quad -1 < \alpha < 1; \tag{1.7}$$

or

$$\sup_{t \in (0,T)} \| \, |x - \bar{x}|^{\alpha} u(x,t) \|_{L_x^{\frac{n}{1-\alpha}}} < \varepsilon, \quad -1 \leq \alpha \leq 1; \tag{1.8}$$

with ε sufficiently small. Then $(0,T) \times \{\bar{x}\}$ is a regular set.

Remark 1.1. The condition $\frac{2}{s} + \frac{n}{p} = 1 - \alpha$ makes the norm

$$\| \, |x - \bar{x}|^{\alpha} u(x,t) \|_{L^s_T L^p_x}$$

scaling invariant with respect to

$$u(t, x - \bar{x}) \to \lambda u(\lambda^2 t, \lambda(x - \bar{x})).$$

Our goal is to point out the local aspect of Theorem 1.2: for each $t \in (0,T)$ there is a neighborhood[1] $\Omega_{t,\bar{x}}$ of $\bar{x}$ such that u is smooth in $\{t\} \times \Omega_{t,\bar{x}}$.

The restriction to a neighborhood of $\bar{x}$ can be heuristically explained in the case $\alpha < 0$: the weight morally localizes the norm around $\bar{x}$ and a loss of information at infinity occurs.

We shall show how to recover this information by a suitable amount of angular regularity (if $\alpha < 0$) and how to do the same in the case $\alpha > 0$ even if weaker angular regularity is assumed.

By translations it is possible to restrict to the case $\bar{x} = 0$. All the following results are of course true provided with the norms and weights centered at $\bar{x} \neq 0$.

[1] We mean a neighborhood in the space variables for each fixed time.