Nonlinear Analysis 105 (2014) 41–49

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Variable exponent Bergman spaces

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ARTICLE INFO

Article history:
Received 19 March 2014
Accepted 2 April 2014
Communicated by Enzo Mitidieri

MSC:
30H20
46E30

Keywords:
Bergman spaces
Nonstandard function spaces
Variable exponent Lebesgue spaces
Bergman projection
Berezin transform

ABSTRACT

In this article we define variable exponent Bergman spaces and show that polynomials are dense in the spaces. We also show that the Bergman projection and the Berezin transform are bounded in these spaces.

1. Introduction

Variable Lebesgue spaces are a generalization of Lebesgue spaces where we allow the exponent to be a measurable function and thus the exponent may vary. It seems that the first occurrence in the literature is in the 1932 paper of Orlicz [1]. The seminal work on this field is the 1991 paper of Kováčik and Rákosník [2] where many basic properties of Lebesgue and Sobolev spaces were shown. To see a more detailed history of such spaces see, e.g., [3, Section 1.1]. These variable exponent function spaces have a wide variety of applications, e.g., in the modeling of electrorheological fluids [4–7] as well as thermorheological fluids [8], in the study of image processing [9–15] and in differential equations and minimization problems with non-standard growth [16–18]. For details on variable Lebesgue spaces one can refer to [19,3,2] and the references therein.

Let $\mathbb{D}$ denote the open unit disk in the complex plane and $\mathrm{d}A$ the normalized Lebesgue measure on $\mathbb{D}$. For a given $1 \leqslant p < \infty$ define the *Bergman space* $A^p(\mathbb{D})$ as the space of all analytic functions on $\mathbb{D}$ that satisfy:

$$\|f\|_{A^p}^p := \int_{\mathbb{D}} |f(z)|^p \mathrm{d}A(z) < \infty.$$

The theory of Bergman spaces was introduced by S. Bergman in [20] and since the 1990s it has gained a lot of attention mainly due to some major breakthroughs at the time. For details on the theory of Bergman spaces we refer to the books [21,22] and the references therein.

In this article we will define variable exponent Bergman spaces and show some fundamental properties. We consider this to be an interesting topic since the classical approach to Bergman spaces seems to fail in the variable framework.

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http://dx.doi.org/10.1016/j.na.2014.04.001

To circumvent this problem, we rely on techniques from real harmonic analysis, variable exponent spaces and complex function theory. The article is distributed as follows. In Section 2 we give some basic notions on variable exponent Lebesgue spaces that will be used. We also define variable exponent Bergman spaces and show that under suitable conditions on the exponent, they are Banach spaces. In Section 3, to deal with the problem of approximation in the variable exponent Bergman spaces, we introduce the concept of *mollified dilations* and show some of its properties. In Section 4 we study the Bergman projection and show it remains bounded in the case of variable exponent Bergman spaces. We also address the problem of duality in this setting.

For the rest of the paper, we will use the notation $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leqslant Cb$. Similarly, we use $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

2. Basic notions

2.1. On Lebesgue spaces with variable exponent

The basics on variable Lebesgue spaces may be found in the monographs [19,3], but we recall here some necessary definitions and propositions. For $\Omega \subset \mathbb{R}^d$ we put $p_\Omega^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x)$ and $p_\Omega^- := \operatorname{ess\,inf}_{x\in\Omega} p(x)$; we use the abbreviations $p^+ = p_\Omega^+$ and $p^- = p_\Omega^-$ when there is no danger of confusion. For a measurable function $p : \Omega \to [1, \infty)$, we call it a *variable exponent*, and define the set of all variable exponents with $p^+ < \infty$ as $\mathscr{P}(\Omega)$.

For a complex-valued measurable function $\varphi : \Omega \to \mathbb{C}$ we define the *modular* $\rho_{p(\cdot)}$ by

$$\rho_{p(\cdot)}(\varphi) := \int_\Omega |\varphi(x)|^{p(x)}\, \mathrm{d}x$$

and the *Luxemburg–Nakano norm* by

$$\|\varphi\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{\varphi}{\lambda}\right) \leqslant 1\right\}. \tag{1}$$

Definition 2.1. Let $p \in \mathscr{P}(\Omega)$. The *variable Lebesgue space* $L^{p(\cdot)}(\Omega)$ is introduced as the set of all complex-valued measurable functions $\varphi : \Omega \to \mathbb{C}$ for which the modular is finite, i.e. $\rho_{p(\cdot)}(\varphi) < \infty$. Equipped with the Luxemburg–Nakano norm (1) this is a Banach space.

Proposition 2.2 (*Hölder's inequality, See Theorem 2.26 in [19]*)**.** *Let $p \in \mathscr{P}(\Omega)$, $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, where $1/p'(x) + 1/p(x) = 1$. Then $fg \in L^1(\Omega)$ and*

$$\int_\Omega |f(x)g(x)|\mathrm{d}x \leqslant 2\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)}.$$

Proposition 2.3 (*Theorem 2.80 in [19]*)**.** *Let $p \in \mathscr{P}(\Omega)$, then the dual space to $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$ (up to an isomorphism), where $1/p'(x) + 1/p(x) = 1$.*

It is known, see [19, Theorem 2.34], that the Luxemburg–Nakano norm (1) of $L^{p(\cdot)}(\Omega)$ is equivalent to the following norm

$$|||f|||_{L^{p(\cdot)}(\Omega)} = \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)}\leqslant 1} \int_\Omega f(x)g(x)\mathrm{d}x,$$

i.e.

$$|||f|||_{L^{p(\cdot)}(\Omega)} \sim \|f\|_{L^{p(\cdot)}(\Omega)}. \tag{2}$$

We need to impose some regularity in the variable exponent in order to have some "fruitful" theory (e.g. the boundedness of the maximal operator).

Definition 2.4. A function $p : \Omega \to \mathbb{R}$ is said to be *locally* log-*Hölder continuous* on Ω if there exists a positive constant C such that

$$|p(x) - p(y)| \leqslant \frac{C}{\log(\mathrm{e} + 1/|x-y|)}, \tag{3}$$

for all $x, y \in \Omega$. We denote by $\mathscr{P}^{\log}(\Omega)$ the set of all locally log-Hölder continuous functions in Ω for which $1 < p_- \leqslant p_+ < \infty$.

One essential tool that we will use is the so-called maximal operator.

Definition 2.5. Given a function $f \in L^1_{\mathrm{loc}}(\Omega)$, the *Hardy–Littlewood maximal function* of f, denote by Mf, is defined for any $x \in \mathbb{R}^n$ by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\mathrm{d}y. \tag{4}$$