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On quasi-periodic solutions for a generalized Boussinesq equation

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In this paper a one-dimensional generalized Boussinesq equation

$$u_{tt} - u_{xx} + (u^3 + u_{xx})_{xx} = 0$$

with hinged boundary conditions is considered. It is proved that the above equation admits small-amplitude quasi-periodic solutions corresponding to finite dimensional invariant tori of an associated infinite dimensional Hamiltonian system. The proof is based on an infinite dimensional KAM theorem and the Birkhoff normal form.

1. Introduction

In 1870s, Boussinesq first explained the phenomena of Scott Russell's solitary wave mathematically [1–3]. By motivation of Euler's equation for two-dimensional, inviscid, irrotational flow beneath a free surface, Boussinesq introduced approximations appropriate for long waves of small amplitude and derived the well-known Boussinesq equation:

$$u_{tt} - u_{xx} - \frac{3}{2}\epsilon(u^2)_{xx} - \frac{\epsilon}{3}u_{xxxx} = 0, \quad x \in \mathbf{R}. \tag{BQ}$$

Eq. (BQ) describes waves moving basically in one direction, for which $u_t + u_x = O(\epsilon)$, and it gives a satisfactory description of steady long waves of small amplitude. The most interesting feature of this equation is that it possesses solitary wave solutions and admits an associated inverse scattering formalism [4–6]. One could find that the dispersion relation $\omega^2 = k^2 - \frac{1}{3}\epsilon k^4$ leads to an unbounded growth rate for high frequency and the initial value problem is linearly ill posed. Instead of this classical Boussinesq equation, some authors considered a variant [7]:

$$u_{tt} - u_{xx} + u_{xxxx} + f(u)_{xx} = 0, \quad x \in \mathbf{R}. \tag{BQ1}$$

Linear plane waves $e^{ikx - i\omega t}$ of Eq. (BQ1) have the dispersion relation $\omega^2 = k^2 + k^4$, this ensures that the initial value problem is linearly well posed. When $f(u) = u^2$, Zakharov [6] derived Eq. (BQ1) as a model of a nonlinear string. Helfrich and Pedlodky [8] obtained the same equation when they applied asymptotic time-dependent theory for coherent structures to a marginally stable baroclinic zonal flow. Moreover, Eq. (BQ1) also describes a model in the study of phase transition in

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shape-memory alloys [9] with the nonlinearity $f(u) = 4u^3 - 6u^5$, and appears in the study of anharmonic lattice waves [10] with a more general polynomial nonlinearity f.

These important models draw many authors' attentions to (BQ1) and there have been many significant results. For example, Bona and Sachs [7] proved the global existence of smooth solutions and stability of solitary waves for (BQ1) with $f(u) = u^2$. For general nonlinearities f, also see Refs. [11–16].

We note that in the previous papers the spatial variable $x \in \mathbf{R}$. In this paper, we restrict x to the interval $[0, \pi]$ and consider the 1-D generalized Boussinesq equation (BQ1) with $f(u) = u^3$ under the hinged boundary conditions:

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} + f(u)_{xx} = 0 \\ u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0. \end{cases} \tag{1.1}$$

It is well known that the problem (1.1) of a generalized Boussinesq equation possesses Hamiltonian structure. Moreover, with the hinged boundary conditions of (1.1) the linear operator $Lu = -u_{xx} + u_{xxxx}$ only has discrete spectrum with some asymptotically increasing property. This important information reminds us of KAM theory for an infinite dimensional Hamiltonian system.

Kuksin [17] first proposed an infinite dimensional KAM Theorem, which can be applied to some PDEs such as 1-D wave equations and 1-D Schrödinger equations. Later, many well known KAM Theorems are obtained for Hamiltonian PDEs; see [18–27] and references therein. We note that the previous papers concerned simple normal frequencies. In 2000, Chierchia and You [28] developed a method for multiple normal frequencies and applied it to 1D wave equations under the periodic boundary condition. Then, by means of decay property and momentum conservation, Geng and You [29] proved a KAM theorem for Hamiltonian PDEs in higher dimensional space. Later, Eliasson and Kuksin [21] proposed the Töplitz–Lipschitz structure, by which they successfully gave an infinite dimensional KAM theorem for higher dimensional Schrödinger equations. A similar property named Quasi-Töplitz was observed by Procesi and Xu [30].

As we know, to make the KAM machine work fluently, a parameter family is required. However, many Hamiltonian PDEs either have no outer parameters or do not have enough. A powerful tool to explore the parameters for these PDEs is the normal form method. This idea was first introduced by Kuksin and Pöschel [31,23]. Then, with this technique, many important results are obtained [32,33,27,34].

An alternative method is developed by Craig, Wayne and Bourgain, usually called the C–W–B method, which is based on Lyapunov–Schmidt decomposition [35–40]. However, this method does not provide any information on the stability of corresponding invariant tori like the KAM method.

The rest of this paper is organized as follows. In Section 2 we reduce (1.1) to an infinite dimensional Hamiltonian system and state our main result. Then we discuss the regularity of Hamiltonian vector field in Section 3 and obtain the Birkhoff normal form in Section 4. In Section 5 we state a well known Cantor manifold theorem and then use it to prove our main result in Section 6. In the Appendix some preliminary results on Poisson bracket are given.

2. Hamiltonian structure and the main result

We first formally introduce the Hamiltonian structure of (1.1). Let $u_t = v_x$, Eq. (1.1) is equivalent to

$$\begin{cases} u_t = v_x \\ v_t = \partial_x(u - u_{xx} - f(u)). \end{cases} \tag{2.1}$$

Let $\mathcal{P} = H_0^1([0, \pi]) \times L^2([0, \pi]),\ g(u) = \int_0^u f(s)ds$, it is easy to see

$$H(w) = \int_0^\pi \frac{u^2}{2} + \frac{v^2}{2} + \frac{u_x^2}{2} - g(u)dx$$

is well defined for $w = (u, v) \in \mathcal{P}$ with $u \in H_0^1([0, \pi]),\ v \in L^2([0, \pi])$.

Let

$$\mathbb{J} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}$$

be a weak derivative operator with respect to the L^2 inner product on the space $L^2([0, \pi]) \times L^2([0, \pi])$. It is defined in the following sense:
for $w(x) = (u(x), v(x)) \in L^2([0, \pi]) \times L^2([0, \pi])$, we define

$$(\mathbb{J}w, z) = -\int_0^\pi \langle w, \mathbb{J}z\rangle\, dx = -\int_0^\pi v(x)\phi'(x) + u(x)\psi'(x)dx,$$

for any $z(x) = (\phi(x), \psi(x)) \in C_0^\infty(0, \pi) \times C_0^\infty(0, \pi)$.

Denote by $\nabla_w H$ the weak derivative of H with respect to the L^2-inner product. Then Eq. (2.1) can be written in a more compact form

$$\frac{dw}{dt} = \mathbb{J}\nabla_w H. \tag{2.2}$$