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On the degenerate hyperbolic Goursat problem for linear and nonlinear equations of Tricomi type

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ABSTRACT

For linear and semilinear equations of Tricomi type, existence, uniqueness and qualitative properties of weak solutions to the degenerate hyperbolic Goursat problem on characteristic triangles will be established. For the linear problem, a robust L^2-based theory will be developed, including well-posedness, elements of a spectral theory, partial regularity results and maximum and comparison principles. For the nonlinear problem, existence of weak solutions with nonlinearities of unlimited polynomial growth at infinity will be proven by combining standard topological methods of nonlinear analysis with the linear theory developed here. For homogeneous *supercritical* nonlinearities, the uniqueness of the trivial solution in the class of weak solutions will be established by combining suitable Pohožaev-type identities with well tailored mollifying procedures. For the linear problem, the weak existence theory presented here will also be connected to known explicit representation formulas for sufficiently regular solutions with the aid of the partial regularity results. For the nonlinear problem, the question what constitutes critical growth for the problem will be clarified and differences with equations of mixed elliptic–hyperbolic type will be exhibited.

1. Introduction

In this work, we will study the existence and uniqueness of *weak solutions u* for the semilinear Goursat[1] problem

$$\begin{cases} Tu = f(x,y,u) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = AC \cup AB, \end{cases} \tag{1.1}$$

where $T \equiv -y\partial_x^2 - \partial_y^2$ is the Tricomi operator on $\mathbb{R}^2$ with cartesian coordinates (x, y), f is a nonlinearity to be specified and $\Omega = ABC$ is a *characteristic triangle*; that is, a simply connected region in the plane whose boundary consists of the segment

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[1] Often called the (first) Darboux problem in the Russian literature.

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AB of the x-axis and the two characteristics arcs AC and BC (of negative and positive slope respectively) that issue from A and B and intersect at C. The solutions will be found in the subspace of the standard Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$ of elements having zero trace on Γ. For the linear problem, where $f(x, y, s) = f(x, y) + \lambda s$ with $\lambda \in \mathbb{R}$, we will develop a robust L^2-based theory whose compact solution operator provides some spectral information and is compatible with weak maximum and comparison principles, which are obtained with the aid of some regularity theory. Then, using standard topological tools of nonlinear analysis and carefully constructed mollifying procedures, we will establish results on existence and uniqueness for the nonlinear problem under suitable hypotheses on the nonlinearity f.

Our primary interest in the nonlinear version (1.1) of the well studied linear Goursat problem is purely mathematical as a companion to our study of analogous questions on mixed type domains (i.e. Ω intersects also the region where $y > 0$). The existence of *weak solutions* with Tricomi boundary conditions has been treated in [1] and with Dirichlet conditions in [2–4] while the uniqueness of *classical solutions* has been treated for various class of domains and boundary conditions [5–8]. More precisely, we will seek to clarify the interaction between the form of the boundary conditions (i.e. Dirichlet conditions on the entire boundary or on a suitable proper subset of the boundary), the geometry of the domain at the parabolic boundary points (i.e. A and B, where the operator degenerates), the regularity of the solutions for $f \in L^2(\Omega)$ (i.e. the presence or not of a weight in the $H^1(\Omega)$ norm of the weak solutions), the resulting barriers to p-summability coming from the Sobolev embedding theorem and its relation to critical exponent phenomena.

We will see that the geometry of having corners in A and B combined with placing the Dirichlet conditions only on the proper subset $\Gamma = AB \cup AC$ allows for weak solutions in $H^1(\Omega)$ and hence no barrier to immersion in $L^p(\Omega)$ for each $p \in [1, +\infty)$. For the mixed type Tricomi problem in *angular domains* (with corners in A and B), the weak solutions also lie in $H^1(\Omega)$, as was shown in [9,1]. On the other hand, both for Tricomi problem in *normal domains* (where the elliptic boundary is orthogonal to the x-axis in A and B) and for the Dirichlet problem on suitable domain, the weak solutions carry the weight $|y|$ on the first derivative in x, as was shown in [10,2]. As a result, one has a critical exponent in the Sobolev embedding which is $2^*(1, 1) = 10$, as noted in [5]. Moreover, the Goursat problem will be shown to admit maximum and comparison principles for weak solutions such as those in the mixed type setting of the Tricomi problem in normal domains [10]; however, for weak solutions with weights as noted above in the mixed type case. The better regularity of the solutions in the Goursat case allows us to apply monotone methods (upper and lower solutions) with no limit on the polynomial growth in s for the nonlinearity $f(x, y, s)$ in contrast to the strong restrictions on growth required in the mixed type setting, as one knows from [1].

In addition, a nonhomogeneous dilation invariance in the Tricomi operator T is known to yield a Pohožaev-type result on the nonexistence of nontrivial solutions u with homogeneous boundary conditions $u = 0$ placed on a large enough portion of the boundary of a suitably star-shaped domains; that is, if $f(x, y, s) = s|s|^{p-2}$ with $p > 10$ and then the only $C^2(\overline{\Omega})$ solutions must vanish identically, as shown in [5,6]. By exploiting the special geometry of the Goursat domain through well-tailored mollifying operators and by exploiting the absence of weights in the weak solutions, we will close the regularity gap between $C^2(\overline{\Omega})$ (where one had uniqueness) and $H^1(\Omega)$ (where one has existence results). Closing this regularity gap was the original motivation for studying the nonlinear Goursat problem (1.1), but much more has come out of the investigation. In particular, with respect to what may constitute critical growth for the problem (1.1), the following situation emerges. There is no polynomial growth barrier for the purposes of existence and no polynomial critical growth exponent for the Sobolev embedding for the weak solutions with no weights. This dissimilarity with elliptic problems should be perhaps explained by the fact that the Goursat problem is not variational. On the other hand, if one were to impose the boundary condition also on the characteristic BC, the problem becomes variational but the extra boundary data forces the weak solutions to carry weights (as in the mixed type Dirichlet problem [2]), which in turn yields the critical exponent for the Sobolev immersion and a probable barrier to existence for weak solutions at supercritical growth. Moreover, the Dirichlet problem loses the maximum principle and hence also the possibility for monotone methods which are used here to solve the superlinear problems.

The plan of the paper is as follows. In Section 2, we recall the necessary machinery and develop the linear solvability and spectral theory. In Section 3, we examine the question of regularity of the weak solutions and maximum/comparison principles compatible with the solvability theory. An important byproduct will be the bridging of a possible gap between *generalized solutions* as given by explicit integral representations involving hypergeometric functions and our notion of weak solutions whose existence follows from suitable *a priori* estimates (see Theorem 3.2 and the discussion in Step 1 of the proof). In Section 4, we prove results on existence of weak solutions. We exploit the contraction mapping principle and Leray–Schauder principles for sublinear and asymptotically linear nonlinearities and monotone methods for superlinear nonlinearities. In Section 5, we prove the aforementioned extension of the uniqueness of the trivial solution for weak solutions. In addition there are two appendices which give the proofs of two technical lemmas concerning the compactness on $C^0(\overline{\Omega})$ of the linear solution operator and the weak maximum principle for regular solutions.

We conclude this introduction with a few additional remarks on the problems considered herein. Some of the results continue to hold for operators of Tricomi type where the coefficient y in the Tricomi operator T is replaced by $K(y)$ which has the sign of y; for example, results on solvability and maximum principles for regular solutions. On the other hand, the maximum principle for weak solutions (and hence the monotone methods of Section 5) makes use of the regularity result of Nakhushev [11] which holds for $K(y) = (-y)^m$ for $m < 2$. We have treated problems in only two dimensions. In part, this is due to the importance of the Tricomi equation in the context of two dimensional transonic potential flow (see the modern survey of Morawetz [12]), but it should be noted that the analogous boundary value problem in higher dimensions (the so-called *Protter problem*) has a solvability theory which is much more delicate (see [13] for example). On the other hand, questions of nonexistence for weak solutions to nonlinear degenerate hyperbolic Cauchy problems in general dimensions