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Fourth-order nonlinear elliptic equations with lower order term and natural growth conditions

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ABSTRACT

We prove the existence of weak solutions of the homogeneous Dirichlet problem related to a class of nonlinear elliptic equations whose prototype is

$$\sum_{|\alpha|=2} \mathrm{D}^\alpha\Big[\mid \mathrm{D}^2 u\mid^{p-2} \mathrm{D}^\alpha u\Big] - \sum_{|\alpha|=1} \mathrm{D}^\alpha\Big[\mid \mathrm{D}^1 u\mid^{q-2} \mathrm{D}^\alpha u\Big] + u\Big[\mid \mathrm{D}^1 u\mid^q + \mid \mathrm{D}^2 u\mid^p\Big] = f$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 3)$ with sufficiently smooth boundary, $u : \Omega \to \mathbb{R}$ is the unknown function, $\mathrm{D}^h u = \Big\{\mathrm{D}^\alpha u :\mid \alpha \mid= h\Big\}$, $\mid \mathrm{D}^h u \mid= \Big[\sum_{|\alpha|=h} \mid \mathrm{D}^\alpha u\mid^2\Big]^{\frac{1}{2}}$, for $h = 1, 2$, numbers $p, q \in [2, N[$ and $f \in L^1(\Omega)$.

0. Introduction

In this paper we prove the existence of weak solutions of the homogeneous Dirichlet problem related to a class of nonlinear elliptic equations whose prototype is

$$\sum_{|\alpha|=2} \mathrm{D}^\alpha\Big[|\mathrm{D}^2 u|^{p-2}\mathrm{D}^\alpha u\Big] - \sum_{|\alpha|=1} \mathrm{D}^\alpha\Big[|\mathrm{D}^1 u|^{q-2}\mathrm{D}^\alpha u\Big] + u\Big[|\mathrm{D}^1 u|^q + |\mathrm{D}^2 u|^p\Big] = f$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 3)$ with sufficiently smooth boundary, $u : \Omega \to \mathbb{R}$ is the unknown function, $\mathrm{D}^h u = \Big\{\mathrm{D}^\alpha u : |\alpha| = h\Big\}$, $|\mathrm{D}^h u| = \big[\sum_{|\alpha|=h} |\mathrm{D}^\alpha u|^2\big]^{\frac{1}{2}}$, for $h = 1, 2$, numbers $p, q \in [2, N[$ and $f \in L^1(\Omega)$.

Although we are restricting ourselves to a fourth-order equation, our method can be extended to higher-order equations.

It is well known that from the point of view of the existence and the regularity properties of a (suitably defined) solution, a high-order equation, under standard structural conditions, behaves like a system of equations. That is, even if the right-hand side f and the boundary datum were regular, say L^∞, an equation of the type

$$\sum_{|\alpha|=1,2} (-1)^{|\alpha|}\mathrm{D}^\alpha A_\alpha(x, \mathrm{D}^1 u, \mathrm{D}^2 u) = f \tag{0.1}$$

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satisfying the usual ellipticity condition

$$\sum_{|\alpha|=2} A_\alpha(x,\xi)\xi_\alpha \geq \mu_1 \sum_{|\alpha|=2} |\xi_\alpha|^p - \mu_2 \sum_{|\alpha|<2} |\xi_\alpha|^{r_\alpha} \tag{0.2}$$

with suitable r_α, could fail to have L^∞-solutions (see [1–7], see also [8]).

Specifically, in 1978 I.V. Skrypnik in [1] introduced a subclass of nonlinear higher order equations having $C^{0,\alpha}$-solutions under the following ellipticity condition

$$\sum_{|\alpha|=1,2} A_\alpha(x,\xi)\xi_\alpha \geq \left[\nu_1 \sum_{|\alpha|=2} |\xi_\alpha|^p + \nu_2 \sum_{|\alpha|=1} |\xi_\alpha|^q\right], \quad q > 2p. \tag{0.3}$$

It is worthwhile to note that condition (0.3) gives back the uniform ellipticity condition in the case of second order equations.

The aforementioned condition, although stronger than condition (0.2), is optimal in the sense that whenever $q \leq 2p$ or the constant ν_2 is replaced by a nonnegative function $\nu_2(x)$ vanishing at at least one point, a weak solution of the homogeneous Dirichlet problem related to Eq. (0.1) could fail to be bounded (see [1] for two counterexamples).

For related arguments on elliptic systems with special structural conditions see also [9–12].

Operators satisfying condition (0.3) have been studied in connection with many other questions such as homogenization problems, qualitative properties of the solutions, removable singularities in the degenerate and non-degenerate cases, in [13–18].

More recently, in [19] A.A. Kovalevsky has considered Eq. (0.1) under the Skrypnik ellipticity condition (0.3) and with L^1-right-hand side and has proved the existence (and uniqueness) of the so called entropy solution, i.e. a solution which satisfies an additional condition similar to the one introduced in the paper [20], but suitably modified in order to fit an high order equation. In particular such an entropy solution belongs to the Sobolev space $W^{1,\lambda}(\Omega) \cap W_0^{2,\mu}(\Omega)$, where $\lambda \in]1, r[$, $\mu \in \left[1, \frac{p}{q}r\right[$, $r = \frac{n(q-1)}{n-1}$, provided $q > \max\left\{\frac{3n-2}{n+p-1}p, \frac{np}{n(p-1)+1}\right\}$.

In this paper we assume that the ellipticity condition (0.3) holds and we prove that the problem

$$\begin{cases} \displaystyle\sum_{|\alpha|=1,2} (-1)^{|\alpha|} \mathrm{D}^\alpha A_\alpha(x, \mathrm{D}^1 u, \mathrm{D}^2 u) + g(x, u, \mathrm{D}^1 u, \mathrm{D}^2 u) = f & \text{in } \Omega \\ \mathrm{D}^\alpha u = 0, \qquad |\alpha| \leq 1 & \text{on } \partial\Omega, \end{cases}$$

where g is a lower order term which behaves as $u\left[|\mathrm{D}^1 u|^q + |\mathrm{D}^2 u|^p\right]$ and $f \in \mathrm{L}^1(\Omega)$, has standard weak solutions, that is solutions which belong to the Sobolev space $\mathrm{W}^{1,q}(\Omega) \cap \mathrm{W}_0^{2,p}(\Omega)$. This means that, as in the case of second order equations, the lower order term has a regularizing effect on the properties of the solution.

As a by-product of our investigation we will prove that if the right-hand side is sufficiently regular, say $f \in L^t(\Omega)$ with $t > n/q$, then there exist bounded weak solutions of the problem (1.8).

One of the main difficulties when dealing with higher order equations is that the standard truncations $T_k(s) = s - [|s| - k]_+ \mathrm{sign}(s)$, $k > 0$, $s \in \mathbb{R}$ cannot be used as test functions, due to the lack of the existence of second derivatives. Thus, to obtain the required estimates in the case of fourth-order equations, we have to construct a class of functions suitably replacing the previous truncations and choose the energy spaces for the approximating problems such that, on the one hand, the superpositions of these new functions and solutions can be differentiated twice within the energy space chosen and, on the other hand, when we use these superpositions as test functions, we can control the remaining terms containing their second derivatives in the corresponding integral identities.

The paper is organized as follows: in the next section we introduce the notations and we state our main results; some auxiliary lemmas are stated in Section 1 as well; Section 3 is devoted to the proof of the existence of a bounded solution when the datum f is regular; finally in Section 4 we prove the existence of weak solutions when $f \in \mathrm{L}^1$.

1. Hypotheses

Let Ω be an open bounded set in $\mathbb{R}^N$ with $N \geq 3$. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$ with nonnegative integer components we denote

$$|\alpha| = \alpha_1 + \cdots + \alpha_N, \quad \mathrm{D}^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}},$$

$$\mathrm{D}^1 u(x) = \left\{\mathrm{D}^\alpha u(x) : |\alpha| = 1\right\}, \quad |\mathrm{D}^1 u(x)|^2 = \sum_{|\alpha|=1} |\mathrm{D}^\alpha u(x)|^2,$$

$$\mathrm{D}^2 u(x) = \left\{\mathrm{D}^\alpha u(x) : |\alpha| = 2\right\}, \quad |\mathrm{D}^2 u(x)|^2 = \sum_{|\alpha|=2} |\mathrm{D}^\alpha u(x)|^2.$$

We denote by $N(2)$ the number of different multi-indices α such that $|\alpha| \leq 2$ and we set $\xi = \left\{\xi_\alpha \in \mathbb{R} : |\alpha| \leq 2\right\} \in \mathbb{R}^{N(2)}$.