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Symphonic join of maps between the spheres

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ABSTRACT

In the previous papers Nakauchi and Takenaka (2011) and Kawai and Nakauchi (2011), the first author introduced a stationary map for the functional of L^2-norm of pullbacks of metrics. We call it a symphonic map. In this paper using the argument of Xu and Yang (1993) (see also Smith (1975) and Ding (1988)), we give a notion of the symphonic join of two symphonic maps between the standard spheres.

1. Introduction

Let (M, g) and (N, h) be Riemannian manifolds without boundary, and let f be a smooth map from M into N. Assume that M is compact. In the previous papers [1,2], the first author introduced the functional of the L^2-norm of pullbacks of metrics

$$\mathcal{F}(f) = \int_M \left\| f^*h \right\|^2 dv_g,$$

where f^*h denotes the pullback of the metric h by f, i.e.,

$$(f^*h)(X, Y) = h\big(df(X), df(Y)\big)$$

for any vector fields X, Y on M, and dv_g is the volume form on (M, g).

The functional $\mathcal{F}(f)$ is related to the energy $E(f)$ in the theory of harmonic maps (See Eells and Lemaire [3] and [4].). In viewpoint of pullbacks of metrics, the energy $E(f)$ can be regarded as the integral of the *trace* of the pullback f^*h, while the functional $\mathcal{F}$ is the integral of the *norm* of the pullback. We consider smooth maps which are critical points of the functional $\mathcal{F}(f)$. We call them *symphonic maps* in this paper. In [1] (See also [5,6].), the first author gave the first and the second variation formulas, the monotonicity formula and a Bochner type formula for symphonic maps. In this paper, using the arguments of Smith [7] and Xu–Yang [8], we give a notion of the symphonic join of two symphonic maps between the spheres.

Let $\big(\mathbb{S}^{n_1}(r_1), g_{\mathbb{S}^{n_1}(r_1)}\big)$ and $\big(\mathbb{S}^{n_2}(r_2), g_{\mathbb{S}^{n_2}(r_2)}\big)$ be the standard spheres of radii r_1 and r_2 respectively. We define a Riemannian join of these two spheres

$$\big(\mathbb{S}^{n_1}(R_1), g_{\mathbb{S}^{n_1}(R_1)}\big) * \big(\mathbb{S}^{n_2}(R_2), g_{\mathbb{S}^{n_2}(R_2)}\big) = \left(\left(\mathbb{S}^{n_1}(R_1) \times \mathbb{S}^{n_2}(R_2) \times \left[0, \frac{\pi}{2}\right]\right) \Big/ \sim, \; g_{\mathbb{S}^{n_1}(R_1) * \mathbb{S}^{n_2}(R_2)}\right)$$

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where

$$(x,\ y_1,\ 0) \sim (x,\ y_2,\ 0),$$
$$\left(x_1,\ y,\ \frac{\pi}{2}\right) \sim \left(x_2,\ y,\ \frac{\pi}{2}\right)$$

for any $x, x_1, x_2 \in \mathbb{S}^{n_1}(R_1)$ and for any $y, y_1, y_2 \in \mathbb{S}^{n_2}(R_2)$, and

$$g_{\mathbb{S}^{n_1}(R_1)*\mathbb{S}^{n_2}(R_2)} = \frac{\cos^2 t}{R_1^2}\, g_{\mathbb{S}^{n_1}(R_1)} + \frac{\sin^2 t}{R_2^2}\, g_{\mathbb{S}^{n_2}(R_2)} + g_{[0,\ \frac{\pi}{2}]}.$$

For simplicity, we denote the join $\left(\mathbb{S}^{n_1}(R_1),\ g_{\mathbb{S}^{n_1}(R_1)}\right) * \left(\mathbb{S}^{n_2}(R_2),\ g_{\mathbb{S}^{n_2}(R_2)}\right)$ by $\mathbb{S}^{n_1}(R_1) * \mathbb{S}^{n_2}(R_2)$. We can verify that the join $\mathbb{S}^{n_1}(R_1) * \mathbb{S}^{n_2}(R_2)$ is isometric to the standard sphere $\mathbb{S}^{n_1+n_2+1}(1)$.

Let

$$f_1 : \mathbb{S}^{m_1}(r_1) \longrightarrow \mathbb{S}^{n_1}(1)$$
$$f_2 : \mathbb{S}^{m_2}(r_2) \longrightarrow \mathbb{S}^{n_2}(1)$$

be smooth maps. Note that the target spheres are of radius 1. Then we consider the *join* of f_1 and f_2

$$f_1 * f_2 : \mathbb{S}^{m_1}(r_1) * \mathbb{S}^{m_2}(r_2) \longrightarrow \mathbb{S}^{n_1}(1) * \mathbb{S}^{n_2}(1)$$

defined by

$$(f_1 * f_2)(x,\ y,\ t) = \big(f_1(x),\ f_2(y),\ \varphi(t)\big)$$

where φ is a given smooth map on $\left[0,\ \frac{\pi}{2}\right]$ satisfying $\varphi(0) = 0$ and $\varphi\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$. Then we have

Theorem 1. *The following two conditions are equivalent:*

(a) *The join $f_1 * f_2$ is a symphonic map.*
(b) *The function φ satisfies the equation*

$$\begin{aligned}&\left((\varphi')^3\right)' + \left(m_1\frac{\cos t}{\sin t} - m_2\frac{\sin t}{\cos t}\right)(\varphi')^3\\ &\quad -\ \sin\varphi\cos\varphi\left(r_1^4\,\frac{\cos^2\varphi(t)}{\cos^4 t}\,\left\|f_1^* g_{\mathbb{S}^{n_1}(1)}\right\|^2 - r_2^4\,\frac{\sin^2\varphi(t)}{\sin^4 t}\,\left\|f_2^* g_{\mathbb{S}^{n_2}(1)}\right\|^2\right) = 0.\end{aligned}$$

We can find a smooth solution $\varphi(t)$ of the ordinary equation (7) to have the following existence theorem.

Theorem 2. *For any two symphonic maps f_1 and f_2, there exists a symphonic join $f_1 * f_2$ of maps f_1 and f_2, i.e., a symphonic map which is a join of these two maps.*

Since the join of the spheres $\mathbb{S}^{m_1}(1)$ and $\mathbb{S}^{m_2}(1)$ is isometric to the standard sphere $\mathbb{S}^{m_1+m_2+1}(1)$, Theorem 2 directly implies the following theorem.

Theorem 3. *If there exist two symphonic maps*

$$f_1 : S^{m_1}(r_1) \longrightarrow S^{n_1}(1)$$
$$f_2 : S^{m_2}(r_2) \longrightarrow S^{n_2}(1),$$

then there exists a symphonic map

$$f_3 : S^{m_1+m_2+1}(1) \longrightarrow S^{n_1+n_2+1}(1).$$

2. Symphonic maps into the sphere

Let $(M,\ g)$, $(N,\ h)$ be Riemannian manifolds without boundary, and let f be a smooth map from M into N. Assume that M is compact. Then we consider the functional

$$\mathcal{F}(f) = \int_M \left\|f^*h\right\|^2 dv_g,$$

where f^*h denotes the pullback of the metric h by f, i.e.,

$$(f^*h)(X,\ Y) = h\big(df(X),\ df(Y)\big)$$

for any vector fields X, Y on M, and dv_g is the volume form on $(M,\ g)$.