Nonlinear Analysis 108 (2014) 128–143

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Moment estimate and existence for solutions of stochastic functional differential equations

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ARTICLE INFO

Communicated by S. Carl

Keywords:
Stochastic functional differential equations
Existence and uniqueness
Stochastic differential inequalities
Stability
Impulses
Equations in Hilbert spaces

ABSTRACT

In this paper, we give the existence–uniqueness theorems and the moment estimates of solutions for a large class of SFDEs. These estimates improve and extend some related results including exponential stability, decay stability and asymptotic behavior. Their corollaries improve and extend the classical Halanay inequality and some of its generalizations. Moreover, the stochastic version of the Wintner theorem in continuous function space is established by the compare principle, which improves and extends the main results of Xu et al. (2008, 2013). When the methods presented are applied to the SFDEs with impulses and SFDEs in Hilbert spaces, we extend the related results of Govindan and Ahmed (2013), Liu et al. (2007, 2010), Vinodkumar (2010) and Xu et al. (2012). Two examples are provided to illustrate the effectiveness of our results.

1. Introduction

Stochastic functional differential equations (SFDEs) play a very important role in formulation and analysis in control engineering, electrical, mechanical and physical sciences, economic and social sciences. Therefore, the theory of SFDEs has been developed very quickly [1–19].

Moment estimate is the most basic and useful technique of analyzing dynamic behavior of SFDEs, which is widely used in studying stability [9], boundedness [13], invariant and attracting properties [3,4,19]. Most results on this topic presuppose the existence of the solution of SFDEs. Therefore, moment estimate and existence have been two of the most popular topics in the study of SFDEs and many interesting results have been obtained [1–19]. In general, the existing results on moment estimate and existence of SFDEs require (i) the coefficients of the SFDEs obey the local Lipschitz condition and the linear growth condition [2,3,5,6,8,10–12,15]; (ii) the Itô operator $\mathcal{L}V$ of energy function (or Lyapunov function) V is bounded by a linear function $q(t) - p(t)V$, where p and q are constants [7,9,17] or $q(t) \equiv 0$ and $p(t)$ is the derivative of logarithm of power function of a decay function [14, Theorem 2.1]. Similarly, the analogous requirements appear in the impulsive SFDEs [11,12] and SFDEs in Hilbert spaces [2,5,7,16]. However, there are many non-autonomous SFDEs which do not obey the local Lipschitz condition and the linear growth condition. Moreover, even if the linear SFDEs have the coefficients of elementary functions, an energy function with the above properties may not be found (see Example 6.1). Hence the existing criteria on moment estimate and existence for SFDEs are not applicable and we see the necessity to develop new criteria. One of our

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http://dx.doi.org/10.1016/j.na.2014.05.004
0362-546X/

aims in this paper is to establish new criteria where the local Lipschitz condition and the linear growth condition are no longer needed while the up-bound for the Itô operator of the energy function may take a much more general form.

On the other hand, to the best of our knowledge, there are few works reported on the global existence of impulsive SFDEs induced by the existence of local solution since most of the existing results for SFDEs need the continuity of their initial functions [10]. However, the local solution obtained in the impulsive SFDEs may be discontinuous so that it cannot become the initial function of the next section of the solution. Although Liu et al. [12] permit the initial functions are PC-valued random variables, their results on the global existence still need the global linear growth condition. Especially, there is not any result for the moment estimate and existence of the strong solutions of impulsive SFDEs in Hilbert spaces. Moreover, although the Halanay inequality [20] has been well developed for the stability of deterministic functional differential equations, so far there is almost no result of Halanay inequality on the stability of SFDEs. The other aims of this paper are to close these gaps.

Motivated by the above works, in this paper, we will give some new results on the moment estimate and existence of solutions for SFDEs with discontinuous initial functions. Our approaches are based on the Razumikhin technique, the compare principle and establishing new Itô operator inequalities. These methods and techniques will be applied to the impulsive SFDEs and the SFDEs in Hilbert spaces. The obtained results extend the classical Halanay inequality [21] and its generalizations [20], and generalize some related works in [7,11,13–18]. Two examples are provided to illustrate the effectiveness of our results.

2. Preliminaries

In this section, we recall some notations and basic definitions and introduce some lemmas.

Let $R^n_+ = \{x \in R^n : x_i \geq 0,\ \forall i = 1, \ldots, n\}$. $M(X; Y)$ and $C(X; Y)$ denote the spaces of Borel measurable and continuous mappings from the topological space X to the topological space Y, respectively. For $-\infty < a < b < \infty$, we say that a function $\phi : [a, b] \to R^n$ is piecewise continuous ($p.c.$ for short) if $\phi(t)$ has at most a finite number of jump discontinuities on $[a, b)$ and $\phi(t^-) = \phi(t)$ for all points in $(a, b]$. $PC([a, b]; R^n)$ denotes the family of piecewise continuous functions from $[a, b]$ to R^n [11,12,22]. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, P)$ stands for a complete probability space and $\omega(t) = (\omega_1(t), \ldots, \omega_m(t))^T$ is an m-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, P)$. A stochastic process $x(t) = x(t, \omega)$ is said to be measurable if the function $x : [a, b] \times \Omega \to R^n$ is measurable [6]. It is said to be continuous or $p.c.$ if for almost all $\omega \in \Omega$ function $x(t)$ is continuous or $p.c.$ on $[a, b]$. It is said to be adapted if for each $t \in [a, b]$, $x(t)$ is $\mathcal{F}_t$-measurable. Let $\tau \geq 0$ be a fixed number or $\tau = \infty$. When $\tau = \infty$, we mean that $[-\tau, 0] = (-\infty, 0]$. Then, we give the following notations.

$PC^b = \{f | f \in PC([-\tau, 0]; R^n)$ and $f(t)$ is bounded on $[-\tau, 0]\}$,

$C^b = \{f | f \in C([-\tau, 0]; R^n)$ and $f(t)$ is bounded on $[-\tau, 0]\}$,

$L^p(J; R^d) = \left\{ f | f \in M(J; R^d),\ \int_J |f(t)|^p dt < \infty \right\}, \quad p > 0,\ J \subseteq R,\ d = n$ or $d = m \times n$,

$\mathcal{M}^p(J; R^d)$: class of R^d-valued measurable $\mathcal{F}_t$-adapted processes $\{f(t)\}_{t\in J}$ such that $\mathbb{E} \int_J |f(t)|^p dt < \infty$ a.s.,

$\mathcal{PC}_{\mathcal{F}_t}(J; R^n)$: class of $\mathcal{F}_t$-adapted processes $\{f(t)\}_{t\in J}$ and $f \in PC(J; R^n)$ a.s.,

$\mathcal{C}_{\mathcal{F}_t}(J; R^n)$: class of $\mathcal{F}_t$-adapted processes $\{f(t)\}_{t\in J}$ and $f \in C(J; R^n)$ a.s.,

$PC^p_{\mathcal{F}_t}$: class of $\mathcal{F}_t$-measurable PC^b-valued random variables ξ_t and $\mathbb{E}\|\xi_t\|^p < \infty$,

$C^p_{\mathcal{F}_t}$: class of $\mathcal{F}_t$-measurable C^b-valued random variables ξ_t and $\mathbb{E}\|\xi_t\|^p < \infty$,

where $\xi_t = \xi(t + \theta)$ for $\theta \in [-\tau, 0]$ and $\|\phi\| = \sup_{-\tau\leq s\leq 0} |\phi(s)|$ is the norm of the spaces C^b or PC^b, where $|\cdot|$ is any norm in R^n or the trace norm of a matrix, i.e., $|A| = \sqrt{\mathrm{Tr}(A^T A)}$.

Let $C^{1,2}(R \times R^n; R)$ denote the family of all functions $V(t, x)$ on $R \times R^n$, which are twice continuously differentiable in x and once in t. For each $V(t, x) \in C^{1,2}(R \times R^n; R)$, we define an Itô operator $\mathcal{L}V(t, x)$ of the function $V(t, x)$, associated with the SFDE (2.2), by

$$\mathcal{L}V(t, x) = V_t(t, x) + V_x(t, x)B(t, x_t) + \frac{1}{2}\mathrm{Tr}[\sigma^T(t, x_t)V_{xx}(t, x)\sigma(t, x_t)]. \tag{2.1}$$

We now consider the following retarded SFDE

$$dx(t) = B(t, x_t)dt + \sigma(t, x_t)d\omega(t), \quad t \in [t_0, T),\ x_{t_0}(s) = \xi(s), \tag{2.2}$$

where T is a constant, or $T = \infty$, $x_t(s) = x(t + s)$, $s \in [-\tau, 0]$ can be regarded as a C^b or PC^b-valued stochastic process. Generally, $x \in R^n$ is called the state variable, $B(t, x_t)$ the drift coefficient and $\sigma(t, x_t)$ the diffusion coefficient.