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Critical exponent for the Cauchy problem to the weakly coupled damped wave system

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ABSTRACT

In this paper, we consider a system of weakly coupled semilinear damped wave equations. We determine the critical exponent for any space dimensions. Our proof of the global existence of solutions for supercritical nonlinearities is based on a weighted energy method, whose multiplier is appropriately modified in the case where one of the exponent of the nonlinear term is less than the so called Fujita's critical exponent. We also give estimates of the lifespan of solutions from above for subcritical nonlinearities.

1. Introduction

This paper is concerned with the Cauchy problem of the weakly coupled system of damped wave equations

$$\begin{cases} u_{tt} - \Delta u + u_t = g(v), & (t,x) \in (0,\infty) \times \mathbf{R}^N, \\ v_{tt} - \Delta v + v_t = f(u), & \\ (u, u_t, v, v_t)(0,x) = \varepsilon(u_0, u_1, v_0, v_1)(x), & x \in \mathbf{R}^N. \end{cases} \tag{1.1}$$

Here u and v are real-valued unknown functions, (u_0, u_1, v_0, v_1) are given initial data and $\varepsilon > 0$ is a small parameter. The functions $g(v)$ and $f(u)$ denote nonlinear terms. The typical examples of (f, g) are

$$(f(u), g(v)) = (|u|^{q-1}u, |v|^{p-1}v), \quad (|u|^q, |v|^p)$$

with $p, q \geq 1$, $pq > 1$.

It is well known that the solutions to the damped wave equation behave like those of the heat equation as time goes to infinity. This is called *diffusion phenomena* and investigated for a long time by many mathematicians (see, for example, [1–7] and the references therein). In this paper, we study the structure of the system (1.1) from the viewpoint of the diffusion phenomena. In particular, we shall prove that if

$$\alpha := \max\left\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\right\} < \frac{N}{2}, \tag{1.2}$$

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then there exists a unique global-in-time solution (u, v) to (1.1) for sufficiently small initial data. Moreover, in the case where $(f(u), g(v)) = (|u|^q, |v|^p)$ and $\alpha > N/2$, we give the estimate of lifespan from above.

We mention some previous results. Eq. (1.1) is closely related to the system of heat equations

$$\begin{cases} u_t - \Delta u = g(v), & (t,x) \in (0,\infty) \times \mathbf{R}^N, \\ v_t - \Delta v = f(u), & \\ (u,v)(0,x) = \varepsilon(u_0, v_0)(x), & x \in \mathbf{R}^N. \end{cases} \tag{1.3}$$

Escobedo and Herrero [8] proved for (1.3) that when the data (u_0, v_0) is nonnegative, bounded continuous and $(f(u), g(v)) = (u^q, v^p)$ with $p, q > 0$, $pq > 1$, the exponents p, q satisfying

$$\alpha := \max\left\{ \frac{p+1}{pq-1}, \frac{q+1}{pq-1} \right\} = \frac{N}{2} \tag{1.4}$$

are critical. Here the "critical" means that (i) if $\alpha < N/2$ (supercritical), then the local-in-time solution can be extended globally for suitably small data and (ii) if $\alpha \geq N/2$ (critical or subcritical), then every local-in-time solution blows up in finite time. We remark that they also proved the existence of global solutions when $0 < pq \leq 1$ without smallness assumption on the data and the blow-up of solutions for large data when $pq > 1$, $\alpha < N/2$.

For (1.1), Sun and Wang [9] proved the existence of global-in-time solutions to (1.1) for suitably small initial data when $N = 1$ or $N = 3$ and $(f(u), g(v)) = (|u|^q, |v|^p)$, $\alpha < N/2$, (for the case $N = 2$, see Narazaki [10]). Moreover, they proved that for any $N \geq 1$, if $\alpha \geq N/2$ and the initial data have positive integral values, then the global-in-time solution of (1.1) does not exist.

Ogawa and Takeda [11] (see also Takeda [12]) considered more general strongly coupled system of damped wave equations

$$u_{tt} - \Delta u + u_t = F(u), \tag{1.5}$$

where $u = {}^t(u_1, \dots, u_k)$, $F(u) = {}^t(F_1(u), \dots, F_k(u))$ and

$$F_j(u) = \prod_{l=1}^{k} |u_l|^{p_{j,l}}.$$

Moreover, put $P = (p_{j,l})_{1 \leq j,l \leq k}$, $\alpha = {}^t(\alpha_1, \dots, \alpha_k) = (P - I)^{-1} \cdot {}^t(1, \dots, 1)$, and $\alpha_{\max} = \max(\alpha_1, \dots, \alpha_k)$, where I denotes the identity matrix. When $N \leq 3$, they proved that if $p_{j,l} \in [1, \infty) \cup \{0\}$, $\sum_{l=1}^{k} p_{j,l} > 1$, $\det(P - I) \neq 0$ and $0 < \alpha_j < N/2$, then the system (1.5) admits a unique global solution for suitably small data. Moreover, they obtained blow-up results for $\max_{1 \leq j \leq k} \alpha_j \geq N/2$ and all $N \geq 1$. In [13] they also obtained the decay estimates of global solutions and that the asymptotic profile is given by a constant multiple of the Gaussian under the condition

$$\min_{1 \leq j \leq k} \sum_{l=1}^{k} p_{j,l} > 1 + 2/N.$$

The first author [14] considered the asymptotic behavior of solutions to (1.1) and (1.3) with small data and the nonlinear term $(f(u), g(v))$ satisfying $f(0) = g(0) = 0$ and

$$|f(u) - f(\tilde{u})| \leq C(|u| + |\tilde{u}|)^{q-1}|u - \tilde{u}|, \qquad |g(v) - g(\tilde{v})| \leq C(|v| + |\tilde{v}|)^{p-1}|v - \tilde{v}|.$$

He remarked that the supercritical condition $\alpha < N/2$ is equivalent to

$$q\left(p - \frac{2}{N}\right) > 1 + \frac{2}{N} =: \rho_F(N). \tag{1.6}$$

Here we note that we may assume that $p \leq q$ without loss of generality. The exponent $\rho_F(N)$ is known as Fujita's critical exponent for the single semilinear heat equation

$$v_t - \Delta v = v^p \tag{1.7}$$

(see [15,16]), that is, if $p > \rho_F(N)$, then the global-in-time solution to (1.7) exists for small data; if $1 < p \leq \rho_F(N)$, then the local-in-time solution with positive data blows up in finite time even for small data (see [17–19,4,6,20,5,3,21], etc. for corresponding results to the single damped wave equation).

In [14], he assorted the supercritical case (1.6) to the following two cases:

$$q \geq p > \rho_F(N), \tag{Case I}$$

$$q > \rho_F(N) \geq p > \frac{2}{N}. \tag{Case II}$$

For (Case I), he proved that the solution (u, v) of (1.3) with small data $\varepsilon(u_0, v_0) \in L^1 \cap L^\infty$ behaves as

$$\|(u - \theta_1 G, v - \theta_2 G)\|_{L^r} = o(t^{\frac{N}{2}\left(1 - \frac{1}{r}\right)})$$

as $t \to +\infty$ for any $1 \leq r \leq \infty$, where θ_1, θ_2 are suitable constants and G is the Gaussian $G(t,x) = (4\pi t)^{-N/2} \exp(-|x|^2/(4t))$. In Case II, to guess the behavior of solutions, he gave the following observation, which is also a foundation for our