Nonlinear Analysis 108 (2014) 275–290

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Uniqueness and non-degeneracy of positive radial solutions for quasilinear elliptic equations with exponential nonlinearity

Shinji Adachi [a,*], Tatsuya Watanabe [b]

[a] Department of Mathematical and Systems Engineering, Faculty of Engineering, Shizuoka University, 3-5-1 Johoku, Naka-ku, Hamamatsu, 432-8561, Japan

[b] Department of Mathematics, Faculty of Science, Kyoto Sangyo University, Motoyama, Kamigamo, Kita-ku, Kyoto-City, 603-8555, Japan

ARTICLE INFO

Article history:
Received 5 November 2013
Accepted 21 May 2014
Communicated by S. Carl

MSC:
35J62
35A02

Keywords:
Quasilinear elliptic equation
Exponential nonlinearity
Uniqueness and non-degeneracy of positive solutions

ABSTRACT

We are concerned with the uniqueness and non-degeneracy result of positive solutions for a class of quasilinear elliptic equations with exponential nonlinearity. We convert a quasilinear elliptic equation into a semilinear one and study precise correspondences between two equations. We show the uniqueness and the non-degeneracy of positive solutions by analyzing the converted semilinear problem.

1. Introduction

In this paper, we consider the following quasilinear elliptic equation with exponential nonlinearity in R:

$$-\Delta u + \lambda u - \kappa \Delta(|u|^2)u = e^u - 1 \quad \text{in } R^2, \tag{1.1}$$

where $\lambda > 1$ and $\kappa > 0$. Our purpose is to show the uniqueness and the non-degeneracy of the positive radial solution of (1.1).

Eq. (1.1) appears in the study of the following modified Schrödinger equation:

$$i\frac{\partial z}{\partial t} = -\Delta z - \kappa \Delta(|z|^2)z - \tilde{h}(|z|^2)z, \quad (t, x) \in (0, \infty) \times R^N, \tag{1.2}$$

(see [1,2] for the derivations and physical backgrounds.) Here we suppose that z is a complex-valued function. Considering the standing wave of the form $z(t, x) = u(x)e^{i\lambda t}$ $(\lambda > 0)$, then we are led to the elliptic problem:

$$-\Delta u + \lambda u - \kappa \Delta(|u|^2)u = h(u) \quad \text{in } R^N, \tag{1.3}$$

where $h(s) = \tilde{h}(|s|^2)s$.

* Corresponding author. Tel.: +81 053 478 1643.
E-mail address: tsadati@ipc.shizuoka.ac.jp (S. Adachi).

http://dx.doi.org/10.1016/j.na.2014.05.020
0362-546X/

There has been many works on the existence of a positive solution of (1.3). See [3–9] and references therein. Especially, a positive solution of (1.3) exists for a wide class of nonlinearities.

On the other hand, the uniqueness and the non-degeneracy of positive solutions are less studied. Moreover we have to restrict the class of nonlinearities. When $N \geq 3$, the authors studied the case $h(s) = |s|^{p-1}s$, $1 < p < \frac{3N+2}{N-2}$ and showed that the positive solution of (1.3) is unique if $\kappa\lambda^{\frac{2}{p-1}}$ is sufficiently large. (See [3].) In [10,11], they proved the uniqueness and non-degeneracy if κ is sufficiently small by applying the perturbation method. In [12], they studied the problem in R^2 and obtained the uniqueness and the non-degeneracy when $h(s)$ grows exponentially. However their argument is also based on the perturbation method and hence their result holds only for small κ.

In this paper, we consider the case $h(s) = e^s - 1$ and obtain the uniqueness and the non-degeneracy of the positive radial solution for another range of parameters λ and κ, which is not covered in the previous papers. Indeed, we have the following result.

Theorem 1.1. *There exists $\lambda^* > 0$ independent of $\kappa > 0$ such that for any $\lambda > \lambda^*$ and $\kappa > 0$, the following properties hold.*

(i) *(1.1) has a unique positive radial solution w.*
(ii) *The kernel of the linearized operator L around w is given by*

$$\mathrm{Ker}(L) = \mathrm{span}\left\{\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}\right\},$$

where the operator $L : H^2(R) \to L^2(R)$ is given by

$$L(\phi) = -(1+2\kappa w^2)\Delta\phi + (\lambda - e^w)\phi - 4\kappa w\nabla w\cdot\nabla\phi - (4\kappa w\Delta w + 2\kappa|\nabla w|^2)\phi. \tag{1.4}$$

Especially, w is non-degenerate in $H^1_{rad}(R) = \{u \in H^1(R); u(x) = u(|x|)\}$.

For applications, we are interested in *ground states* of (1.1). We define the energy functional $J : X \to R$ and the function space X associated to (1.1) by

$$J(u) = \frac{1}{2}\int_R (|\nabla u|^2 + \lambda u^2)\,dx + \kappa\int_R |\nabla u|^2u^2\,dx - \int_R (e^u - 1 - u)\,dx,$$

$$X = \{u \in H^1(R^2);\ \int_R |\nabla u|^2u^2\,dx < \infty\}.$$

Then a solution w of (1.1) is said to be a ground state of (1.1) if it satisfies

$$J(w) = \inf\{J(u)\ ;\ J'(u) = 0, u \in X \setminus \{0\}\}.$$

As we will see in Proposition 2.8, any ground state of (1.1) is positive and radially symmetric. Thus by Theorem 1.1, we have the following corollary. We believe that this result is important for applications, for example, the stability of the standing wave solution.

Corollary 1.2. *Under the assumption of Theorem 1.1, the ground state of (1.1) is unique and non-degenerate in $H^1_{rad}(R)$.*

To prove Theorem 1.1, we adapt *dual approach* as in [3,4,7]. More precisely, we convert our quasilinear equation into a semilinear one by using a suitable translation f. We will see that there is a one-to-one correspondence between our quasilinear problem and the converted semilinear problem. This correspondence enables us to apply uniqueness results [13,14] for semilinear elliptic equations.

On the other hand, the above correspondence is not sufficient to study the non-degeneracy of the unique positive radial solution. As we will see in Lemma 2.3 and Proposition 2.4, there is a strong relation between the linearized operator of the original quasilinear equation and that of the converted semilinear equation. By this relation, we have only to study the non-degeneracy for the converted semilinear problem. We emphasize that Lemma 2.3 and Proposition 2.4 themselves seem to be new and of interest.

This paper is organized as follows. In Section 2, we introduce the dual approach and show the existence of a positive radial solution. In Section 3, we study the uniqueness of the positive radial solution. In Section 4, we prove the non-degeneracy of the unique positive radial solution. Finally we complete the proof of Theorem 1.1 in Section 5.

2. Dual approach and the existence of a positive radial solution

2.1. Properties of the unique solution of the ODE related to (1.1)

In this subsection, we study some properties of the unique solution of the ODE related to (1.1). As we will see later, this unique solution gives a one-to-one correspondence between (1.1) and semilinear elliptic problem which is given in (2.2).