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On the problem of unique continuation for the p-Laplace equation

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ABSTRACT

We study if two different solutions of the p-Laplace equation

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0,$$

where $1 < p < \infty$, can coincide in an open subset of their common domain of definition. We obtain some partial results on this interesting problem.

1. Introduction

We consider the p-Laplace equation in an open connected set $G \subset \mathbb{R}^n$, $n \geq 2$,

$$\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0, \quad 1 < p < \infty. \tag{1.1}$$

For $p = 2$ we recover the Laplace equation $\Delta u = 0$. We study the question whether two different solutions to (1.1) can coincide in an open subset of their common domain of definition G.

This problem of unique continuation is still, to the best of our knowledge, an open problem, except for the planar case $n = 2$. The planar case has been solved by Alessandrini [1], and by a different approach by Manfredi [2] and Bojarski and Iwaniec in [3], as they have observed that the complex gradient of a solution to (1.1) is quasiregular.

In addition, there are some recent partial results of the unique continuation property for the game p-Laplace equation on trees. We refer to [4].

We refrain from giving a detailed bibliographical account on the literature on unique continuation results for linear elliptic equations in divergence form. We refer to the papers [5,6] by Garofalo and Lin, and to a more recent paper by Alessandrini [7], and suggest the reader to consult also their bibliographies for more detailed information on the subject.

In the present paper, we deal with the problem of unique continuation by studying a certain generalization of Almgren's frequency function for the p-Laplacian. Our results, along with the notation and the preliminary results, are stated in Section 2. The proofs can be found in Sections 3–5.

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2. Results

Let G be an open connected subset of $\mathbb{R}^n$. We consider the p-Laplace equation (1.1) in the weak form

$$\int_G |\nabla u|^{p-2}\nabla u \cdot \nabla \eta \, dx = 0, \tag{2.1}$$

where $\eta \in C_0^\infty(G)$ and $1 < p < \infty$. We refer the reader to, e.g., Heinonen et al. [8] and Lindqvist [9] for a detailed study of the p-Laplace equation and various properties of its solutions. We mention in passing, however, that the weak solutions of (1.1) are $C^{1,\alpha}_{\text{loc}}(G)$, where α depends on n and p. We refer to DiBenedetto [10], Lewis [11], and Tolksdorf [12] for this regularity result. Hence, without loss of generality, we may redefine u so that $u \in W^{1,p}_{\text{loc}}(G) \cap C^1(G)$.

Let us define the *frequency function*

$$F_p(r) = \frac{r \int_{B(z,r)} |\nabla u|^p \, dx}{\int_{\partial B(z,r)} |u|^p \, dS}, \tag{2.2}$$

where $\overline{B}(z, r) \subset G$; we denote

$$D(r) = \int_{B(z,r)} |\nabla u|^p \, dx \quad \text{and} \quad I(r) = \int_{\partial B(z,r)} |u|^p \, dS.$$

Observe that $F_p(r)$ is not defined for such radii r for which $I(r) = 0$. We remark that $F_p(r)$ is a generalization of the well known Almgren frequency function

$$F_2(r) = \frac{r \int_{B(z,r)} |\nabla u|^2 \, dx}{\int_{\partial B(z,r)} |u|^2 \, dS} \tag{2.3}$$

for harmonic functions in $\mathbb{R}^n$. To the best of our knowledge, $F_p(r)$, $p \neq 2$, has not been previously studied in the literature. It might be interesting to study other generalizations, for instance, the case in which r is replaced with r^{p-1} in (2.2). We have, however, omitted such considerations here.

The main results of the present paper are the following theorems.

Theorem 2.4. *Suppose $u \in C^1(G)$. Assume further that there exist two concentric balls $B_{r_b} \subset \overline{B}_{R_b} \subset G$ such that the frequency function $F_p(r)$ is defined, i.e., $I(r) > 0$ for every $r \in (r_b, R_b]$, and moreover, $\|F_p\|_{L^\infty((r_b,R_b])} < \infty$. Then there exists some $r^\star \in (r_b, R_b]$ such that*

$$\int_{\partial B_{r_1}} |u|^p \, dS \le 4 \int_{\partial B_{r_2}} |u|^p \, dS, \tag{2.5}$$

for every $r_1, \ r_2 \in (r_b, r^\star]$. In particular, the following weak doubling property is valid

$$\int_{\partial B_{r^\star}} |u|^p \, dS \le 4 \int_{\partial B_r} |u|^p \, dS, \tag{2.6}$$

for every $r \in (r_b, r^\star]$.

In the following we formulate a result which says that the local boundedness of the frequency function implies certain vanishing properties of the solution. In this respect the situation is similar to the linear case $p = 2$, and we thus generalize this phenomenon to every $1 < p < \infty$.

Theorem 2.7. *Suppose u is a solution to the p-Laplace equation in G. Consider arbitrary concentric balls $B_{r_b} \subset \overline{B}_{R_b} \subset G$. Assume the following: whenever $I(r) > 0$ for every $r \in (r_b, R_b]$, then $\|F_p\|_{L^\infty((r_b,R_b])} < \infty$. Then if u vanishes on some open ball in G, u is identically zero in G.*

It remains an open problem whether the frequency function $F_p(r)$ is locally bounded for the solutions to the p-Laplace equation. Local boundedness combined with the method of the present paper would solve the unique continuation problem for Eq. (1.1).

In Section 5 we study the question whether a solution can coincide with an affine function without being identically affine in the whole common domain of definition. Clearly an affine function is a solution to the p-Laplace equation. In Propositions 5.1–5.2 we provide an answer to this question. This is a nonlinear generalization of the corresponding phenomenon known for harmonic functions. Perhaps surprisingly, this feature is rather easy to achieve while the classical unique continuation principle for the p-Laplace equation still remains an open problem.

In Section 6 we discuss some observations which might be of interest for further studies.