Nonlinear Analysis 101 (2014) 124–143

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Global attractors for multivalued semiflows with weak continuity properties

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ARTICLE INFO

Article history:
Received 12 July 2013
Accepted 29 January 2014
Communicated by S. Carl

MSC:
76D05
76F10
76F20
47J20
49J40

Keywords:
Multivalued semiflow
Attractor
Global weak solution
Weak topology
Hemivariational inequality
Reaction–diffusion problem

ABSTRACT

A method is proposed to prove the global attractor existence for multivalued semiflows with weak continuity properties. An application to the reaction–diffusion problems with nonmonotone multivalued semilinear boundary condition and nonmonotone multivalued semilinear source term is presented.

1. Introduction

There are at least three approaches to prove the existence of the global attractor for the problems without the uniqueness of solutions: the method of multivalued semigroups or multivalued semiflows developed in the ground-breaking paper of Babin and Vishik [1] (see also [2]), the method of generalized semiflows (see [3]), and the method of trajectory attractors (see [4–7]). The method of trajectory attractors that was related to the other two ones in [8] relies on the study of shift operators on the sets of time dependent trajectories while the other two approaches, which are discussed in relation to each other in [9], consist in the direct study of the sets of states obtainable from the given initial conditions after some period of time. This mapping, known as multivalued semigroup or multivalued semiflow (m-semiflow) is denoted as $\mathbb{R}^+ \times H \ni (t, x) \to G(t, x) \subset 2^H$, where H is the suitable Banach (or metric) space of the problem states. In order to show the existence of a compact global attractor, i.e. the compact set in H that is invariant (or sometimes only negatively semiinvariant) and attracts all bounded sets in H three properties are required: the existence of a set that is bounded in H and absorbs all trajectories of G after some finite time, some compactness type property of G and some continuity or closedness type property of m-semiflow $x \to G(t, x)$. Classically, this last property is the upper semicontinuity (in the sense

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http://dx.doi.org/10.1016/j.na.2014.01.026
0362-546X/

of multifunctions) with respect to the strong topology in the argument space and strong topology in the value space (see [3,2]). In the work of Zhong, Yang and Sun [10] the approach to show the existence of a global attractor for problems governed by semiflows (i.e. problems with the uniqueness of solutions) which are only strong–weak continuous is presented. This approach was further developed in [11] where some results for nonautonomous strict m-semiflows are shown. For the recent results on dynamical systems governed by parabolic problems with uniqueness of solutions see [12–14].

The present paper is on one hand the extension of the results of [3,2] since the condition of semiflow upper semicontinuity is relaxed to the condition called (NW) in the sequel and on the other hand the extension of works [10,11] to a more general, multivalued case. The motivation for the introduction of this condition is twofold: firstly, as it is shown in Lemmata 3.7 and 3.11 below, it is natural to verify for the problems with multifunctions having the form of Clarke subdifferential since it follows from basic a priori estimates and passing to the limit argument; secondly it can replace the strong–strong upper semicontinuity and graph closedness in the abstract theorems on the attractor existence even if an m-semiflow is only point dissipative and nonstrict (see Corollary 2.2 and Theorem 2.4 below).

Note that in [11] the extension of the approach of [10] to the case of m-semiflows that are strict (i.e. such that $G(t+s, x) = G(t, G(s, x))$ for all $x \in H$ and $s, t \in \mathbb{R}^+$) is proposed, while in the present study we consider the case where only the inclusion $G(t + s, x) \subset G(t, G(s, x))$ is assumed to hold. Moreover we propose another generalization of strong–weak continuity than [11], namely for $x_n \to x$ strongly in H and $\xi_n \in G(t, x_n)$ we assume that ξ_n must have a weakly convergent subsequence while in [11] it is assumed that the whole sequence ξ_n must converge.

Attractors for partial differential equations and inclusions without the uniqueness were studied in the recent articles of Kasyanov [15,16] where the approaches by m-semiflows and trajectory attractors were used for first order autonomous evolution equations and inclusions with general nonlinear pseudomonotone operators. The results were adapted to second order evolution inclusions and hemivariational inequalities in [17,18]. Note, that in [15–18] the strong–strong upper semicontinuity of m-semiflow is always used and the compactness is proved by the analysis of the energy function monotonicity.

Another interesting recent article on the existence of global attractors for m-semiflows is the paper of Coti Zelati [19], where only the strict case is considered and the semiflow closedness is assumed to hold only at some time instant $t^* > 0$ and not for all $t \geq 0$.

Examples presented in the present study show that the condition (NW), that states that the multivalued semiflow has weakly compact values and is strong–weak upper semicontinuous, is natural to check for the problems governed by differential inclusions where the multivalued term has the form of Clarke subdifferential.

For an exhaustive review of recent results on the theory of asymptotic behavior for problems without the uniqueness of solutions see [20]. The difficulty in the analysis of these problems lies in the fact that it remains unknown if every solution can be obtained as the limit of the solutions of approximative problems (for example Galerkin problems) and, in consequence, the estimates that hold for the approximate solutions do not have to hold for all solutions of the original problem (see Section 4.3.1 in [20]). It must be remarked here that while most authors consider only the existence of attractors for the multivalued semiflows, there are almost no results on the attractor properties, like their dimension, the attraction speed or the attractor structure. The notable exceptions are the article of Arrieta et al. [21] where for the one dimensional nonlinear reaction–diffusion problem it is shown that the attractor consists of heteroclinic connections between a countable number of fixed points, the article of Kapustyan et al. [22] where the characterization of an attractor for the problem governed by the nonlinear reaction–diffusion equation by means of stable and unstable manifolds of the rest points is given and the article of Kasyanov et al. [23] where the regularity of all weak solutions and their attractors for reaction–diffusion type evolution inclusions was studied.

The plan of this article is the following: in Section 2 we present abstract results on the attractor existence while in Section 3 we present examples of the problems for which we show the attractor existence by means of the proposed abstract framework.

2. Abstract theory of global attractors for multivalued semiflows with a weak continuity property

Let H be a Banach space, and $P(H)$ be the family of all nonempty subsets of H. Some definitions and results of this section remain valid for more general setup of metric spaces and in such cases it will be explicitly noted that H is only a metric space. By $B(x, r)$ we will denote the closed ball centered in $x \in H$ with the radius $r \in \mathbb{R}^+$. Note that here, and in the sequel of this paper we denote $\mathbb{R}^+ = [0, \infty)$.

If H is a metric space equipped with the metric $\rho(\cdot, \cdot)$, then for $x \in H$ and $B \subset H$, we set $\text{dist}_H(x, B) = \inf_{y \in B} \rho(x, y)$. Moreover if $A, B \subset H$ then we define the Hausdorff semidistance from A to B by $\text{dist}_H(A, B) = \sup_{x \in A} \text{dist}_H(x, B)$. Same definitions are valid for normed spaces with $\rho(x, y)$ replaced by $\|x - y\|$.

Definition 2.1. The map $G : \mathbb{R}^+ \times H \to P(H)$ is called a multivalued semiflow (m-semiflow) if:

(1) $G(0, z) = z$ for all $z \in H$.
(2) $G(t + s, z) \subset G(t, G(s, z))$ for all $z \in H$ and all $t, s \geq 0$.

Definition 2.2. An m-semiflow is strict if $G(t + s, z) = G(t, G(s, z))$ for all $z \in H$ and all $t, s \geq 0$.