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Stability of a turnpike phenomenon for approximate solutions of nonautonomous discrete-time optimal control systems

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ABSTRACT

We study turnpike properties of approximate solutions of nonautonomous discrete-time optimal control systems which are determined by sequences of lower semicontinuous objective functions. To have these properties means that the approximate solutions of the problems are determined mainly by the objective functions, and are essentially independent of the choice of intervals and endpoint conditions, except in regions close to the endpoints. We show that these turnpike properties are stable under small perturbations of the objective functions.

1. Introduction

The study of the existence, the structure and properties of (approximate) solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research [1–15]. These problems arise in engineering [16,17], in models of economic growth [18–24,15,25–27], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [28,29], in the calculus of variations on time scales [30,31] and in the theory of thermodynamical equilibrium for materials [32,33].

In this paper we study the structure of approximate solutions of nonautonomous discrete-time optimal control systems arising in economic dynamics which are determined by sequences of lower semicontinuous objective functions.

For each nonempty set Y denote by $\mathcal{B}(Y)$ the set of all bounded functions $f : Y \to R^1$ and for each $f \in \mathcal{B}(Y)$ set

$$\|f\| = \sup\{|f(y)| : y \in Y\}.$$

For each nonempty compact metric space Y denote by $C(Y)$ the set of all continuous functions $f : Y \to R^1$.

Let (X, ρ) be a compact metric space with the metric ρ. The set $X \times X$ is equipped with the metric ρ_1 defined by

$$\rho_1((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2), \quad (x_1, x_2), (y_1, y_2) \in X \times X.$$

For each integer $t \geq 0$ let Ω_t be a nonempty closed subset of the metric space $X \times X$.

Let $T \geq 0$ be an integer. A sequence $\{x_t\}_{t=T}^{\infty} \subset X$ is called a program if $(x_t, x_{t+1}) \in \Omega_t$ for all integers $t \geq T$.

Let T_1, T_2 be integers such that $0 \leq T_1 < T_2$. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ is called a program if $(x_t, x_{t+1}) \in \Omega_t$ for all integers t satisfying $T_1 \leq t < T_2$.

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We assume that there exists a program $\{x_t\}_{t=0}^{\infty}$. Denote by $\mathcal{M}$ the set of all sequences of functions $\{f_t\}_{t=0}^{\infty}$ such that for each integer $t \ge 0$

$$f_t \in \mathcal{B}(\Omega_t) \tag{1.1}$$

and that

$$\sup\{\|f_t\| : \ t = 0, 1, \ldots\} < \infty. \tag{1.2}$$

For each pair of sequences $\{f_t\}_{t=0}^{\infty}, \{g_t\}_{t=0}^{\infty} \in \mathcal{M}$ set

$$d(\{f_t\}_{t=0}^{\infty}, \{g_t\}_{t=0}^{\infty}) = \sup\{\|f_t - g_t\| : \ t = 0, 1, \ldots\}. \tag{1.3}$$

It is easy to see that $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is a metric on $\mathcal{M}$ and that the metric space $(\mathcal{M}, d)$ is complete.

Let $\{f_t\}_{t=0}^{\infty} \in \mathcal{M}$. We consider the following optimization problems

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \to \min \text{ s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program},$$

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \to \min \text{ s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y,$$

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \to \min \text{ s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y, x_{T_2} = z,$$

where $y, z \in X$ and integers T_1, T_2 satisfy $0 \le T_1 < T_2$.

The interest in these discrete-time optimal problems stems from the study of various optimization problems which can be reduced to this framework, e.g., continuous-time control systems which are represented by ordinary differential equations whose cost integrand contains a discounting factor [20], the study of the discrete Frenkel–Kontorova model related to dislocations in one-dimensional crystals [28,29] and the analysis of a long slender bar of a polymeric material under tension in [32,33]. Similar optimization problems are also considered in mathematical economics [19–21,23,15,25–27]. In [34] these problems were considered in the case when $f_t = f_0$ and $\Omega_t = X \times X$ for all integers $t \ge 0$, in [35,36] they were studied in the case when $\Omega_t = X \times X$ for all integers $t \ge 0$ and in [25–27] we studied these problems in the case when $f_t = f_0$ and $\Omega_t = \Omega_0$ for all integers $t \ge 0$. Here we study a general case when the optimal control system is determined by a nonstationary sequence of objective functions $\{f_t\}_{t=0}^{\infty}$ and by a nonstationary sequence of sets of admissible pairs $\{\Omega_t\}_{t=0}^{\infty}$. This makes the situation more realistic but more difficult and less understood.

For each $y, z \in X$ and each pair of integers T_1, T_2 satisfying $0 \le T_1 < T_2$ set

$$U(\{f_t\}_{t=0}^{\infty}, T_1, T_2) = \inf\left\{\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) : \ \{x_t\}_{t=T_1}^{T_2} \text{ is a program}\right\}, \tag{1.4}$$

$$U(\{f_t\}_{t=0}^{\infty}, T_1, T_2, y) = \inf\left\{\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) : \ \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y\right\}, \tag{1.5}$$

$$U(\{f_t\}_{t=0}^{\infty}, T_1, T_2, y, z) = \inf\left\{\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) : \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y, \ x_{T_2} = z\right\}. \tag{1.6}$$

Here we assume that the infimum over empty set is ∞.

Denote by $\mathcal{M}_{reg}$ the set of all sequences of functions $\{f_i\}_{i=0}^{\infty} \in \mathcal{M}$ for which there exist a program $\{x_t^f\}_{t=0}^{\infty}$ and constants $c_f > 0, \gamma_f > 0$ such that the following conditions hold:

(C1) the function f_t is lower semicontinuous for all integers $t \ge 0$;

(C2) for each pair of integers $T_1 \ge 0, T_2 > T_1$,

$$\sum_{t=T_1}^{T_2-1} f_t(x_t^f, x_{t+1}^f) \le U(\{f_t\}_{t=0}^{\infty}, T_1, T_2) + c_f;$$

(C3) for each $\epsilon > 0$ there exists $\delta > 0$ such that for each integer $t \ge 0$ and each $(x, y) \in \Omega_t$ satisfying $\rho(x, x_t^f) \le \delta$, $\rho(y, x_{t+1}^f) \le \delta$ we have

$$|f_t(x_t^f, x_{t+1}^f) - f_t(x, y)| \le \epsilon;$$