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# Stable representation of convex Hamiltonians

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ABSTRACT

Existence and uniqueness of solutions to a Hamilton–Jacobi equation

$$v_t + H(t, x, v_x) = 0, \qquad v(0, \cdot) = \varphi(\cdot)$$

with $H$ convex with respect to the last variable can be proved by associating to $H$ either a Calculus of Variations or an optimal control problem. The data of the new problem should be so that its Hamiltonian coincides with $H$ and should also inherit appropriate continuity/local Lipschitz continuity properties of $H$. In other words, $H$ can be represented by functions describing an optimization problem. In this paper we provide further developments of representation theorems from Rampazzo (2005). In particular, our results imply continuous dependence of representations on the mapping $H$. We apply them to study existence of solutions to the Hamilton–Jacobi equation with $H$ possibly discontinuous in $t$.



## 1. Introduction

Consider a Hamiltonian $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that is convex in the last variable and the Hamilton–Jacobi equation

$$-v_t + H(t, x, -v_x) = 0, \qquad v(T, \cdot) = \varphi(\cdot). \tag{1}$$

Such way of stating the Cauchy problem is more convenient for our purposes than the usual initial value problem. Replacing $t$ by $T - t$ and redefining $H$, (1) may be reduced to the equation mentioned in the abstract.

Let $H^*(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the Fenchel conjugate of $H(t, x, \cdot)$ and consider the Calculus of Variations problem

$$v(t_0, x_0) = \inf\left\{\varphi(x(T)) + \int_{t_0}^{T} H^*(t, x(t), x'(t))dt \,:\, x \in W^{1,1},\ x(t_0) = x_0\right\},$$

where $W^{1,1}$ denotes the space of absolutely continuous mappings from $[t_0, T]$ into $\mathbb{R}^n$. Under appropriate assumptions, $v$ is the unique (viscosity) solution of (1), see for instance [1]. It may happen however that $H^*$ takes infinite values.

A natural question arises then: can we associate to $H$ a compact set $U$ and mappings $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $l : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ inheriting Lipschitz like regularity properties of $H$ and such that $f(t, x, U)$ is equal to the domain of $H^*(t, x, \cdot)$ and

$$H(t, x, p) = \max_{u \in U}(\langle p, f(t, x, u)\rangle - l(t, x, u)). \tag{2}$$

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That is $H$ is equal to the Hamiltonian of a Bolza optimal control problem:

$$V(t_0, x_0) = \inf \left\{ \varphi(x(T)) + \int_{t_0}^{T} l(t, x(t), u(t))dt \mid (x, u) \in \mathcal{S}(t_0, x_0) \right\}, \tag{3}$$

where $\mathcal{S}(t_0, x_0)$ denotes the set of all trajectory-control pairs of the control system

$$\begin{cases} x'(t) = f\big(t, x(t), u(t)\big), & u(t) \in U \text{ a.e.} \\ x(t_0) = x_0. \end{cases} \tag{4}$$

Under appropriate assumptions, $V$ is the unique solution of (1), cf. [1]. When working with control systems it is usually required from $f$ to be so that to every measurable control $u : [t_0, T] \to U$ corresponds a unique solution $x(\cdot)$ of (4) defined on $[t_0, T]$. This is guaranteed, for instance, by the local Lipschitz continuity and the sublinear growth of $f$ with respect to $x$. Such regularity of $f$ is also very helpful for proving continuity/local Lipschitz continuity of $V$.

Denote by $F(t, x)$ the domain of $H^*(t, x, \cdot)$. Then $F(t, x)$ is convex and it can be parameterized in the way preserving Lipschitz like properties of $F$, see for instance [2].

A couple $(f, l)$ satisfying (2) is called in [3] a faithful representation of the Hamiltonian $H$ whenever $f$ enjoys Lipschitz continuity with respect to $x$. In [3, Theorem 3.2] the moduli of continuity of $H$ with respect to $x$ and $p$ are assumed to be time independent and it is claimed that for every $R > 0$ there exists $M_R \geq 0$ such that $F(t, \cdot)$ is $M_R$-Lipschitz on the ball $B(0, R)$ for any $t$. This is proved by applying a contradiction argument for each fixed $t$ and some additional justifications are needed to make $M_R$ time independent. The interested reader can also find in [3] bibliographical references and comments on the literature concerning representation theorems. An earlier result from [4] proposes a representation involving continuous functions $f, l$ and expresses the solution of a stationary Hamilton–Jacobi equation as the value function of an associated infinite horizon optimal control problem. As the author underlines himself, this may be annoying, because it does not allow to associate with a given control $u(\cdot)$ a unique trajectory $x(\cdot)$ of (4).

In the present paper, assuming that $H(t, \cdot, p)$ is Lipschitz on $B(0, R)$ with a time dependent Lipschitz constant $c_R(t)(1 + |p|)$, we prove the existence of $f, l$ as above such that $f(t, \cdot, u)$ is $10nc_R(t)$-Lipschitz. Furthermore, we provide sufficient conditions for continuous dependence of $f$ and $l$ on $H$ (see Section 4 for more details).

The obtained faithful representation is applied then to study existence of a lower semicontinuous solution to the Hamilton–Jacobi equation (1) for Hamiltonian $H$ merely measurable with respect to time. This solution is given by the value function of the Bolza problem associated to $f, l$. We do not investigate uniqueness of solutions here, though, this can be done in the same vein as in [5].

Let us mention that a representation of solutions of the Hamilton–Jacobi equation (with $H(t, x, \cdot)$ not necessarily convex) using value functions of differential games was proposed for instance in [6], see also the bibliography contained therein. A more narrow convex case considered here allows to exploit recent advances of optimal control theory. In particular, having at disposal representation (2) with $f, l$ depending on $H$ in a continuous way, is the main tool of [7] to investigate stability of solutions under constraints on $x$.

The outline of the paper is as follows. In Section 2 we recall some notions and introduce some notations. In Section 3 we provide a representation theorem and in Section 4 we investigate stability of representations. Finally, in Section 5, existence of lower semicontinuous solutions to the Hamilton–Jacobi equation with $t$-measurable Hamiltonian is proved.

## 2. Preliminaries and notations

The notations $B_R$ and $B(0, R)$ stand for the closed ball in $\mathbb{R}^n$ of center zero and radius $R \geq 0$ and $B := B_1$. We denote by $\langle p, v \rangle$ the scalar product of $p, v \in \mathbb{R}^n$ and by $|p|$ the euclidean norm of $p$. For a map $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ its domain is defined by

$$\text{dom}(\phi) \doteq \{x \in \mathbb{R}^n : |\phi(x)| < +\infty\}$$

and its epigraph epi$(\phi)$ by

$$\text{epi}(\phi) \doteq \{(x, a) : x \in \mathbb{R}^n, \ a \in \mathbb{R}, \ a \geq \phi(x)\} \subseteq \mathbb{R}^{n+1}.$$

If $\phi$ is differentiable at $x \in \mathbb{R}^n$, then $\nabla \phi(x)$ states for its gradient at $x$.

For a nonempty subset $K$ of $\mathbb{R}^n$ define $\|K\| := \sup_{x \in K} |x|$. Recall that the *normal cone* to a convex set $K \subset \mathbb{R}^n$ at $x \in K$ is defined by

$$N_K(x) \doteq \{p \in \mathbb{R}^n : \langle p, y - x \rangle \leq 0, \ \forall\, y \in K\}.$$

For a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $\partial f(x)$ denotes its subdifferential at $x \in \text{dom}(f)$ and $f^*$ its Fenchel conjugate. Recall that $(q, -1) \in N_{\text{epi}(f)}(x, f(x))$ if and only if $q \in \partial f(x)$.

**Lemma 2.1** ([8, p. 476]). *For any proper, lower semicontinuous, convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, one has $\partial f^* = (\partial f)^{-1}$ and $\partial f = (\partial f^*)^{-1}$. In the other words $p \in \partial f^*(v)$ is equivalent to $v \in \partial f(p)$ and is equivalent to $f(p) + f^*(v) = \langle v, p \rangle$.*