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Asymptotic behavior of radially symmetric solutions for the Burgers equation in several space dimensions

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ABSTRACT

We study large-time behavior of the radially symmetric solution for the Burgers equation on the exterior of a ball in a multi dimensional space, where boundary data at the far field are prescribed. Liu et al. (1998) considered the asymptotic behavior of the solution of scalar viscous conservation law for the case where the corresponding Riemann problem for the hyperbolic part admits a rarefaction wave. In the present paper, it is proved that for a radially symmetric solution to the Burgers equation on a multidimensional space, the asymptotic behaviors are the same as in Liu et al. (1998). Furthermore, we also derive the time convergence rate. The proof is given by a standard L^2 energy method and a time weighted energy method.

1. Introduction

We consider the Burgers equation on a multi-dimensional space,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \mu \Delta u, \quad (t > 0,\ x \in \mathbb{R}^n), \tag{1.1}$$

where μ is a positive constant. In this paper, we study a radially symmetric solution for (1.1) on the exterior domain $|x| > r_0$ for some positive constant r_0, where the data on the boundary and at the far field are prescribed. For this purpose, we transform the unknown function $u(t, x)$ in (1.1) to $v(t, r)$ by means of $u \equiv (x/r)v(t, r)$, where r is defined by $r := |x|$. Then we have the initial–boundary value problem for the Burgers equation:

$$\begin{cases} v_t + vv_r = \mu \left(v_{rr} + (n-1)\left(\dfrac{v}{r}\right)_r \right), & r > r_0,\ t > 0, \\ v(t, r_0) = v_-, \qquad \lim\limits_{r\to+\infty} v(t, r) = v_+, & t > 0, \\ v(0, r) = v_0(r), & r > r_0, \end{cases} \tag{1.2}$$

where the initial data v_0 is assumed to satisfy $v_0(r_0) = v_-$ and $\lim_{r\to+\infty} v_0(r) = v_+$ as the compatibility conditions. We are interested in the large time behavior of the solution in the case where the corresponding Riemann problem for the hyperbolic part admits a transonic rarefaction wave.

For viscous conservation laws on the half line, T.-P. Liu, A. Matsumura and K. Nishihara [1] investigated the case where the flux is convex and the corresponding Riemann problem for the hyperbolic part admits a transonic rarefaction wave. It was shown in [1] that depending on the signs of the characteristic speeds, asymptotic states of the solutions are classified into three cases, that is, stationary wave, rarefaction wave and composite wave. On the other hand, the analysis of the large-time behavior of the radially symmetric solution for viscous conservation laws on the multi-dimensional space has been open.

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Initial–boundary problem (1.2) is one of the case studies for this open problem. In the present paper, we show that even for the solution of (1.2), the asymptotic behavior is similar to that of [1], that is, the asymptotic states are divided into three cases dependent on the signs of the characteristic speeds on the boundary and at the far field. We consider the following three cases: (a) $v_- < v_+ \le 0$, (b) $0 = v_- < v_+$ and (c) $v_- < 0 < v_+$.

In case (a), as all characteristic speeds of corresponding Riemann problem are negative, we can expect that the solution tends to a stationary wave ϕ which is defined through the stationary problem corresponding to (1.2),

$$\begin{cases} \left(\frac{1}{2}\phi^2\right)_r = \mu\left(\phi_{rr} + (n-1)\left(\frac{\phi}{r}\right)_r\right), & r > r_0, \\ \phi(r_0) = v_-, \qquad \lim_{r\to+\infty}\phi(r) = v_+, \end{cases} \tag{1.3}$$

and we have the following theorem.

Theorem 1.1. *Suppose* (a) *holds. Assume that $v_0 \in H^1$. Let $\phi(r)$ be the stationary solution satisfying the problem* (1.3)*. Then the initial–boundary value problem* (1.2) *has a unique solution v globally in time satisfying*

$$v - \phi \in C^0([0,\infty); H^1), \qquad (v-\phi)_r \in L^2(0,\infty; H^1),$$

and the asymptotic behavior

$$\lim_{t\to\infty}\sup_{r>r_0}\left|v(r,t) - \phi(r)\right| = 0.$$

In addition to the above, assume $v_0 \in L^1$. Then the solution satisfies the following quantitative estimate:

$$\|(v-\phi)(t)\|_{H^1} \le C(1+t)^{-\frac{1}{4}}.$$

In case (b), as all characteristic speeds of the corresponding Riemann problem are positive, then we can expect that the solution tends to a rarefaction wave ψ^R which is defined by $\psi^R((r-r_0)/t) = \psi^R(s)$ for $t > 0$, where $\psi^R(s)$ is defined by

$$\psi^R(s) = \begin{cases} 0, & s \le 0 (= v_-), \\ s, & 0 \le s \le v_+, \\ v_+, & v_+ \le s. \end{cases} \tag{1.4}$$

We have the following theorem.

Theorem 1.2. *Suppose* (b) *holds. Assume that $v_0 - v_+ \in H^1$. Let ψ^R be the rarefaction wave satisfying* (1.4)*. Then the initial–boundary value problem* (1.2) *has a unique solution v globally in time satisfying*

$$v - v_+ \in C^0([0,\infty); H^1), \qquad (v-v_+)_r \in L^2(0,\infty; H^1),$$

and the asymptotic behavior

$$\lim_{t\to\infty}\sup_{r>r_0}\left|v(r,t) - \psi^R\left(\frac{r-r_0}{t}\right)\right| = 0.$$

In addition to the above, assume $v_0 - v_+ \in L^1$. Then the solution satisfies the following quantitative estimate:

$$\left\|v - \psi^R\left(\frac{\cdot}{t}\right)\right\|_{H^1} \le C(1+t)^{-\frac{1}{4}}\log^2(2+t).$$

We consider case (c). As the characteristic speed of the corresponding Riemann problem changes from negative to positive, we can expect that the solution tends to a superposition of a stationary wave and a rarefaction wave. The statement of the theorem for the case (c) is as follows.

Theorem 1.3. *Suppose* (c) *holds. Assume that $v_0 - v_+ \in H^1$. Let $\phi(r)$ be the stationary wave satisfying problem* (1.3) *and $\psi^R((r-r_0)/t)$ be rarefaction wave defined by* (1.4)*. Then the initial–boundary value problem* (1.2) *has a unique solution v globally in time satisfying*

$$v - v_+ \in C^0([0,\infty); H^1), \qquad (v-v_+)_r \in L^2(0,\infty; H^1),$$

and the asymptotic behavior

$$\lim_{t\to\infty}\sup_{r>r_0}\left|v(r,t) - \phi(r) - \psi^R\left(\frac{r-r_0}{t}\right)\right| = 0.$$

In addition to the above, assume $v_0 - v_+ \in L^1$. Then the solution satisfies the following quantitative estimate:

$$\|(v-\phi-\psi^R)(t)\|_{H^1} \le C(1+t)^{-\frac{1}{4}}\log^2(2+t).$$