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Gradient flows for non-smooth interaction potentials

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ABSTRACT

We deal with a nonlocal interaction equation describing the evolution of a particle density under the effect of a general symmetric pairwise interaction potential, not necessarily in convolution form. We describe the case of a convex (or λ-convex) potential, possibly not smooth at several points, generalizing the results of Carrillo et al. (2011). We also identify the cases in which the dynamic is still governed by the continuity equation with well-characterized nonlocal velocity field.

1. Introduction

Let us consider a distribution of particles, represented by a Borel probability measure μ on $\mathbb{R}^d$. We introduce the interaction potential $\mathbf{W} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. The value $\mathbf{W}(x, y)$ describes the interaction of two particles of unit mass at the positions x and y. The total energy of a distribution μ under the effect of the potential is given by the interaction energy functional, defined by

$$\mathcal{W}(\mu) := \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{W}(x, y)\, d(\mu \times \mu)(x, y). \tag{1.1}$$

We assume that $\mathbf{W}$ satisfies the following assumptions:

(i) $\mathbf{W}$ is symmetric, i.e.

$$\mathbf{W}(x, y) = \mathbf{W}(y, x) \quad \text{for every } x, y \in \mathbb{R}^d; \tag{1.2}$$

(ii) $\mathbf{W}$ is a λ-convex function for some $\lambda \in \mathbb{R}$, i.e.

$$\text{there exists } \lambda \in \mathbb{R} \text{ such that } (x, y) \mapsto \mathbf{W}(x, y) - \frac{\lambda}{2}(|x|^2 + |y|^2) \text{ is convex;} \tag{1.3}$$

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(iii) $\mathbf{W}$ satisfies the quadratic growth condition at infinity, i.e.

$$\text{there exists } C > 0 \text{ such that } \mathbf{W}(x, y) \leq C(1 + |x|^2 + |y|^2) \text{ for every } x, y \in \mathbb{R}^d. \tag{1.4}$$

We are interested in the evolution problem given by the continuity equation

$$\partial_t \mu_t + \operatorname{div}(\boldsymbol{v}_t \mu_t) = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^d, \tag{1.5}$$

describing the dynamics of the particle density μ_t, under the mutual attractive–repulsive interaction described by functional (1.1). For any t, μ_t is a Borel probability measure and the velocity vector field $\boldsymbol{v}_t$ enjoys a nonlocal dependence on μ_t. For instance, in the basic model represented by a C^1 potential $\mathbf{W}$ which depends only on the difference of its variables, so that we may write $\mathbf{W}(x, y) = W(x - y)$, the velocity is given by convolution:

$$\boldsymbol{v}_t = -\nabla W * \mu_t. \tag{1.6}$$

Under the assumptions (1.2)–(1.4) and $\mathbf{W}(x, y) = W(x - y)$, in general W is not differentiable but only subdifferentiable, therefore it is reasonable to consider a velocity field of the form

$$\boldsymbol{v}_t = -\boldsymbol{\eta}_t * \mu_t, \tag{1.7}$$

where, for any t, $\boldsymbol{\eta}_t$ represents a Borel measurable selection in the subdifferential of W, and we will write $\boldsymbol{\eta}_t \in \partial W$. Unlike the case (1.6), in general such selection may depend on t. We stress that, for fixed t, the map $x \mapsto \boldsymbol{\eta}_t(x)$ needs to be pointwise defined, since the solutions we consider are probability measures, and since this model typically presents concentration phenomena when starting with absolutely continuous initial data.

In this paper, we are going to analyze equations of the form (1.5)–(1.7) as the gradient flow of the interaction energy (1.1) in the space of Borel probability measures with finite second moment, endowed with the metric-differential structure induced by the so-called Wasserstein distance. This interpretation coming from the optimal transport theory was introduced in [1,2] for nonlinear diffusion equations and generalized for a large class of functionals including potential, interaction, and internal energy by different authors [3–5], see [6] for related information.

The gradient flow interpretation allows to construct solutions by means of variational schemes based on the euclidean optimal transport distance as originally introduced in [7] for the linear Fokker–Planck equation. The convergence of these variational schemes for general functionals was detailed in [4]. The results in this monograph apply to the interaction equation (1.5)–(1.6), with a C^1 smooth nonnegative potential verifying the convexity assumption (1.3) and a growth condition at infinity weaker than (1.4).

On the other hand, these equations have appeared in the literature as simple models of inelastic interactions [8–12] in which the asymptotic behavior of the equations is given by a total concentration towards a unique Dirac Delta point measure. The typical potential in these models was a power law, $\mathbf{W}(x, y) = |x - y|^\alpha,\ \alpha \geq 0$. Moreover, it was noticed in [11] that the convergence towards this unique steady state was in finite time for certain range of exponents in the one dimensional case.

Also these equations appear in very simplified swarming or population dynamics models for collective motion of individuals, see [13–17] and the references therein. The interaction potential models the long-range attraction and the short-range repulsion typical in animal groups. In case the potential is fully attractive, Eq. (1.5) is usually referred as the aggregation equation. For the aggregation equation, finite time blow-up results for weak-L^p solutions, unique up to the blow-up time, have been obtained in the literature [15,18,19]. In fact, those results conjectured that solutions tend to concentrate and form Dirac Deltas in finite time under suitable conditions on the interaction potential. On the other hand, the confinement of particles is shown to happen for short-range repulsive long-range attractive potentials under certain conditions [20]. Some singular stationary states such as uniform densities on spheres have been identified as stable/unstable for radial perturbations in [17] with sharp conditions on the potential. Finally, in the one dimensional case, stationary states formed by finite number of particles and smooth stationary profiles are found whose stability has been studied in [21,22] in a suitable sense.

A global-in-time well-posedness theory of measure weak solutions has been developed in [23] for interaction potentials of the form $\mathbf{W}(x, y) = W(x-y)$ satisfying the assumptions (1.2)–(1.4), and additionally being C^1-smooth except possibly at the origin. The convexity condition (1.3) restricts the possible singularities of the potential at the origin since it implies that W is Lipschitz, and therefore the possible singularity cannot be worse than $|x|$ locally at the origin. Nevertheless, for a class of potentials in which the local behavior at the origin is like $|x|^\alpha,\ 1 \leq \alpha < 2$, the solutions converge towards a Dirac Delta with the full mass at the center of mass of the solution. The condition for blow-up is more general and related to the Osgood criterion for uniqueness of ODEs [15,23,18]. Note that the center of mass of the solution is preserved, at least formally, due to the symmetry assumption (1.2).

In this work, we push the ideas started in [23] further in the direction of giving conditions on the interaction potential to have a global-in-time well-posedness theory of measure solutions. The solutions constructed in Section 2 will be *gradient flow solutions*, as in [4], built via the variational schemes based on the optimal transport Wasserstein distance. The crucial point for the analysis in this framework is the identification of the velocity field in the continuity equation satisfied by the limiting curve of measures from the approximating variational scheme. In order to identify it, we need to characterize the sub-differential of the functional defined in (1.1) with respect to the differential structure induced by the Wasserstein metric. The Wasserstein sub-differential of the functional $\mathcal{W}$, which is rigorously introduced in Section 2, is defined through