Nonlinear Analysis 99 (2014) 1–15

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On conjugacies between piecewise-smooth circle maps

Habibulla Akhadkulov [a,*], Akhtam Dzhalilov [b], Mohd Salmi Md Noorani [a]

[a] School of Mathematical Sciences, Faculty of Science and Technology, University Kebangsaan Malaysia, 43600 UKM Bangi, Selangor Darul Ehsan, Malaysia

[b] Turin Polytechnic University, Kichik Halka yuli 17, Tashkent 100095, Uzbekistan

ARTICLE INFO

Article history:
Received 25 July 2013
Accepted 21 December 2013
Communicated by Enzo Mitidieri

MSC:
37E10
37C15
37C40

Keywords:
Circle homeomorphism
Break point
Rotation number
Invariant measure
Conjugating map
Singular function

ABSTRACT

Let f_i, $i = 1, 2$, be piecewise C^1 circle homeomorphisms with two break points, $\log Df_i$, $i = 1, 2$, are absolutely continuous on each continuity interval of Df_i and $D \log Df_i \in L^p$ for some $p > 1$. Suppose, the jump ratios of f_1 and f_2 at their break points do not coincide but f_1, f_2 have the same total jumps (i.e. the product of jump ratios) and identical irrational rotation number of bounded type. Then the map h conjugating f_1 and f_2 is a singular function, that is, it is continuous on S^1, but $Dh(x) = 0$ almost everywhere with respect to Lebesgue measure.

1. Introduction

The quotient space $S^1 = \mathbb{R}/\mathbb{Z}$ endowed with clearly defined addition, orientation, metric and Lebesgue measure is called the *unit circle*. Let $\pi : \mathbb{R} \to S^1$ be the corresponding projection mapping that "winds" the straight line $\mathbb{R}$ onto the unit circle S^1. We can lift any orientation preserving circle homeomorphism to an homeomorphism $L_f : \mathbb{R} \to \mathbb{R}$ which is required to satisfy the property $L_f(x+1) = L_f(x) + 1$ and that is connected with f by relation $\pi \circ L_f = f \circ \pi$. This homeomorphism L_f is called a *lift* of the homeomorphism f and is unique up to an additive integer term. The most important arithmetic characteristic of the homeomorphism f of the unit circle S^1 is the *rotation number*

$$\rho(f) = \lim_{i\to\infty} \frac{L_f^i(x)}{i} \mod 1,$$

where L_f is the lift of f. Here and below, for a given map F, F^i denotes its ith iterate. The classical Denjoy's theorem says [1], that if f is a circle diffeomorphism with irrational rotation number $\rho = \rho(f)$ and $\log Df$ is of bounded variation, then f is conjugate to the linear rotation $f_\rho : x \to x + \rho \mod 1$, that is, there exists a unique (up to additive constant) circle homeomorphism φ with $f = \varphi^{-1} \circ f_\rho \circ \varphi$. Since the conjugating map φ and the unique f-invariant probability measure μ_f are related by $\varphi(x) = \mu_f([0, x])$ (see [2]), regularity properties of the conjugating map φ imply corresponding properties of the density of the absolutely continuous f-invariant probability measure μ_f. This problem of smoothness of the conjugacy of smooth diffeomorphisms is now very well understood (see [3–9]).

* Corresponding author. Tel.: +60 1115776747.
E-mail addresses: akhadkulov@yahoo.com, habib.13@list.ru (H. Akhadkulov), a_dzhalilov@yahoo.com (A. Dzhalilov), msn@ukm.my (M.S. Md Noorani).

0362-546X/$ – see front matter © 2014 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.na.2013.12.013

A natural extension of diffeomorphisms of the circle are piecewise-smooth homeomorphisms with break points, that is, homeomorphisms which are smooth everywhere except at some singular points, called break points, having a derivative jump. Notice that Denjoy's result can be extended to circle homeomorphisms with break points. Below we present the exact statement of the corresponding theorem. The regularity properties of invariant measures of such homeomorphisms are quite different from the case of diffeomorphisms. Namely, invariant measures of piecewise-smooth circle homeomorphisms with break points and with irrational rotation number are singular w.r.t. Lebesgue measure (see [10–13]). In this case, the conjugacy φ between f and linear rotation f_ρ is a singular function. Here naturally arises a question regarding regularity of conjugacy between two circle maps with break points. Consider two piecewise-smooth circle homeomorphisms f_1, f_2 which have break points with the same order on the circle and the same irrational rotation numbers. Under what conditions is the conjugacy between two such homeomorphisms smooth? This is the rigidity problem for circle homeomorphisms with break points. Denote by $\sigma_f(b) := Df_-(b)/Df_+(b)$ the *jump ratio* or *jump* of f at the break point b. The case of circle maps with one break point and the same jump ratio were studied in detail by K. Khanin and D. Khmelev [14] and K. Khanin and A. Teplinsky [15]. Let $\rho = 1/(k_1 + 1/(k_2 + \cdots + 1/(k_n + \cdots))) := [k_1, k_2, \ldots, k_n, \ldots)$ be the continued fraction expansion of the irrational rotation number ρ. Define

$$M_o = \{\rho : \exists C > 0, \forall n \in \mathbb{N},\ k_{2n-1} \le C\}, \qquad M_e = \{\rho : \exists C > 0, \forall n \in \mathbb{N},\ k_{2n} \le C\}.$$

We formulate the main result of [15].

Theorem 1.1. *Let $f_i \in C^{2+\alpha}(S^1 \setminus \{b_i\}),\ i = 1, 2, \alpha > 0$ be two circle homeomorphisms with one break point that have the same jump ratio σ and the same irrational rotation number $\rho \in (0, 1)$. In addition, let one of the following restrictions be true: either $\sigma > 1$ and $\rho \in M_e$ or $\sigma < 1$ and $\rho \in M_o$. Then the map h conjugating the homeomorphisms f_1 and f_2 is a C^1-diffeomorphism.*

In the case of homeomorphisms with different jump ratios the following theorem was proved by A. Dzhalilov, H. Akin and S. Temir in [16].

Theorem 1.2. *Let $f_i \in C^{2+\alpha}(S^1 \setminus \{b_i\}),\ i = 1, 2, \alpha > 0$ be two circle homeomorphisms with one break point that have different jump ratio and the same irrational rotation number $\rho \in (0, 1)$. Then the map h conjugating the homeomorphisms f_1 and f_2 is a singular function.*

Now consider two piecewise-smooth circle homeomorphisms f_1 and f_2 with m ($m \ge 2$) break points and the same irrational rotation number. Denote by $BP(f_1)$ and $BP(f_2)$ the sets of break points of f_1 and f_2 respectively.

Definition 1.3. The homeomorphisms $f_1,\ f_2$ are said to be *break equivalent* if there exists a topological conjugacy ψ_0 such that

(1) $\psi_0(BP(f_1)) = BP(f_2)$;
(2) $\sigma_{f_2}(\psi_0(b)) = \sigma_{f_1}(b)$, for all $b \in BP(f_1)$.

The rigidity problem for the break equivalent $C^{2+\alpha}$-homeomorphisms and with trivial total jumps (i.e. it is equal to 1) was studied by K. Cunha and D. Smania in [17]. It was proven that any two such homeomorphisms with some combinatorial conditions are C^1-conjugated. The main idea of this work is to consider piecewise-smooth circle homeomorphisms as generalized interval exchange transformations. The case of non-break equivalent homeomorphisms with two break points was studied by H. Akhadkulov, A. Dzhalilov and D. Mayer in [18]. The main result of [18] is the following theorem.

Theorem 1.4. *Let $f_i \in C^{2+\alpha}(S^1 \setminus \{a_i, b_i\}),\ i = 1, 2$ be circle homeomorphisms with two break points a_i, b_i. Assume that*

(1) *their rotation numbers $\rho(f_i),\ i = 1, 2$ are irrational and coincide i.e. $\rho(f_1) = \rho(f_2) = \rho,\ \rho \in \mathbb{R}^1 \setminus \mathbb{Q}$;*
(2) *there exists a topological conjugacy ψ such that $\psi(BP(f_1)) = BP(f_2)$;*
(3) *$\sigma_{f_1}(a_1)\sigma_{f_1}(b_1) \ne \sigma_{f_2}(a_2)\sigma_{f_2}(b_2)$.*

Then the map h conjugating f_1 and f_2 is a singular function.

Now we consider a wider class of circle homeomorphisms with break points. We say that the circle homeomorphism f with finite number of break points satisfy the generalized conditions of Katznelson–Ornstein (KO), if $\log Df$ is absolutely continuous on each continuity interval of Df and $D \log Df \in L^p$ for some $p > 1$. In this work we study the conjugating map h between two circle homeomorphisms f_1 and f_2 with two break points and satisfying the (KO) conditions. Now we formulate the main result of the present paper.

Theorem 1.5. *Let $f_i,\ i = 1, 2$ be piecewise-smooth C^1 circle homeomorphisms with two break points a_i, b_i. Assume that*

(1) *the rotation numbers $\rho(f_i)$ of $f_i,\ i = 1, 2$ are irrational of bounded type (i.e. the coefficients in the continued fraction expansion of $\rho(f_i)$ are bounded) and coincide;*
(2) *$\sigma_{f_1}(a_1)\sigma_{f_1}(b_1) = \sigma_{f_2}(a_2)\sigma_{f_2}(b_2)$;*
(3) *$\sigma_{f_1}(a_1) \ne \sigma_{f_2}(b)$ for all $b \in BP(f_2)$;*
(4) *the break points of $f_i,\ i = 1, 2$ do not lie on the same orbit;*
(5) *$f_i,\ i = 1, 2$ satisfy (KO) conditions for the same $p > 1$.*

Then the map h conjugating f_1 and f_2 is a singular function.