Nonlinear Analysis 99 (2014) 35–48

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

A note on Kirchhoff-type equations with Hartree-type nonlinearities

Dengfeng Lü *

School of Mathematics and Statistics, Hubei Engineering University, Hubei 432000, PR China

ARTICLE INFO

Article history:
Received 15 October 2013
Accepted 24 December 2013
Communicated by S. Carl

MSC:
35J60
35Q55
35J10

Keywords:
Kirchhoff-type equation
Hartree-type nonlinearity
Variational methods
Ground state solution

ABSTRACT

We consider the following Kirchhoff-type equation in $\mathbb{R}^3$

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+(1+\mu g(x))u=\left(\frac{1}{|x|^\alpha}*|u|^p\right)|u|^{p-2}u,$$

where $a>0$, $b\geq 0$ are constants, $\alpha\in(0,3)$, $p\in(2,6-\alpha)$, $\mu>0$ is a parameter and $g(x)$ is a nonnegative continuous potential satisfying some conditions. By using the Nehari manifold and the concentration compactness principle, we establish the existence of ground state solutions for the equation if the parameter μ is large enough. Moreover, some concentration behaviors of these solutions as $\mu\to+\infty$ are discussed.

1. Introduction and main results

In this paper, we are concerned with the existence and concentration of ground state solutions for the following Kirchhoff-type equation in $\mathbb{R}^3$:

$$\begin{cases}-\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V_\mu(x)u=\left(\dfrac{1}{|x|^\alpha}*|u|^p\right)|u|^{p-2}u,\\ u\in H^1(\mathbb{R}^3),\end{cases}\tag{$\mathcal{P}_\mu$}$$

where $a>0$, $b\geq 0$ are constants, $V_\mu(x)=1+\mu g(x)$, $\mu>0$ is a parameter and $g(x)$ is a continuous potential function on $\mathbb{R}^3$, $\alpha\in(0,3)$, $p\in(2,6-\alpha)$ and $*$ is a notation for the convolution of two functions in $\mathbb{R}^3$.

It is pointed out in [1] that such equations arise in various models of physical and biological systems, for example, problem $(\mathcal{P}_\mu)$ is related to the stationary analogue of the equation

$$\rho\frac{\partial^2u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2u}{\partial x^2}=0,\tag{1.1}$$

which was proposed by Kirchhoff in [2] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model considers the changes in the length of the string produced by transverse vibrations. The

* Fax: +86 712 2345434.
E-mail address: dengfeng@hbeu.edu.cn.

0362-546X/$ – see front matter
http://dx.doi.org/10.1016/j.na.2013.12.022

parameters in Eq. (1.1) have the following meanings: ρ is the mass density, P_0 is the initial tension, h is the area of the cross-section, E is the Young modulus of the material, and L is the length of the string. After the pioneer work of Lions [3], where a functional analysis approach was proposed, problem (1.1) began to call attention of several researchers. In recent years, many authors studied the following Kirchhoff-type equation

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u) & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3),\ u>0 & \text{in } \mathbb{R}^3,\end{cases} \tag{1.2}$$

where f is a $\mathcal{C}^1$ function and satisfies certain conditions. Some interesting results were obtained. For example, in [4], Wu obtained existence results for nontrivial solutions and a sequence of high energy solutions for problem (1.2) by applying the symmetric mountain pass theorem. Subsequently, Liu and He [5] proved the existence of infinitely many high energy solutions for (1.2) when f is a subcritical nonlinearity which needs not satisfy the usual Ambrosetti–Rabinowitz-type growth conditions. We also note that several existence results have been obtained for (1.2) on a bounded domain $\Omega\subset\mathbb{R}^3$, see [1,6–8] and we also refer the reader to [9–13] for more results.

On the other hand, in many physical applications, the Hartree-type nonlinearities appear naturally, that is $f(x,u)=(K(x)*G(u))g(u)$, where $G\in\mathcal{C}^1(\mathbb{R},\mathbb{R})$ and $g=G'$. The function $K(x)$ here is usually called the response function. In $(\mathcal{P}_\mu)$, if we set $a=1$, $b=0$, $\mu=0$, $\alpha=1$ and $p=2$, it reduces to the following equation

$$-\Delta u+u=\left(\frac{1}{|x|}*|u|^2\right)u,\quad u\in H^1(\mathbb{R}^3). \tag{1.3}$$

Eq. (1.3) is usually called the Choquard equation which arises in various branches of mathematical physics, such as the quantum theory of large systems of nonrelativistic bosonic atoms and molecules, physics of multiple-particle systems, etc., see for example [14]. Eq. (1.3) was proposed by Choquard in 1976 as an approximation to Hartree–Fock theory for one component plasma [15]. It was also proposed by Penrose [16] as a model for the self-gravitational collapse of a quantum mechanical wave function. Lieb [15] and Lions [17] obtained the existence of solutions for (1.3) by using variational methods. Further results for related problems may be found in [18–22] and the references therein.

Recently, Ma and Zhao [23] considered the generalized Choquard equation

$$-\Delta u+u=\left(\frac{1}{|x|^\alpha}*|u|^p\right)|u|^{p-2}u,\quad u\in H^1(\mathbb{R}^N), \tag{1.4}$$

where $p\ge 2$. Under some assumptions on N, α and p, they proved that every positive solution of (1.4) is radially symmetric and monotone decreasing about some point. More recently, Clapp and Salazar [24] gave the existence of positive and sign changing solutions of (1.4) when $\mathbb{R}^N$ and u are replaced by bounded domains Ω and $W(x)u$ respectively. Moroz and Schaftingen [25] showed the regularity, positivity and radial symmetry of the ground state solutions for the optimal range of parameters, and also obtained decay asymptotics at infinity for them.

Motivated by the works we mentioned above, in this paper, we study the existence and concentration of ground state solutions for a class of Kirchhoff-type equations involving Hartree-type nonlinearities. Such problems are often referred to as being nonlocal because of the appearance of the terms $(\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u$ and $\int_{\mathbb{R}^3}\left(\frac{1}{|x|^\alpha}*|u|^p\right)|u|^pdx$ which imply that problem $(\mathcal{P}_\mu)$ is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties, which make the study of such problems particularly interesting. To the best of our knowledge, it seems that there is almost no work on the existence of ground state solutions to Eq. $(\mathcal{P}_\mu)$, which is just our aim. The main difficulties when dealing with this problem lie in the presence of the nonlocal terms and the lack of compactness due to the unboundedness of the domain $\mathbb{R}^3$. Different from the results mentioned above, by exploiting the Nehari manifold method and the concentration compactness principle, we get the ground state solutions to problem $(\mathcal{P}_\mu)$ and the asymptotic behavior of these solutions as $\mu\to+\infty$. Before stating our main results, we need to introduce some hypotheses on the potential function $g(x)$:

(g_1) $g(x)\in\mathcal{C}(\mathbb{R}^3,\mathbb{R})$ and $g(x)\ge 0$ for all $x\in\mathbb{R}^3$;
(g_2) $\Omega=\operatorname{int} g^{-1}(0)$ is nonempty with smooth boundary and $\overline{\Omega}=g^{-1}(0)$;
(g_3) there exists $M>0$ such that $\mathcal{L}(\{x\in\mathbb{R}^3|g(x)\le M\})<\infty$, where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^3$.

This kind of hypotheses was first introduced by Bartsch and Wang [26] in the study of a nonlinear Schrödinger equation. The hypotheses (g_1)–(g_3) imply that $V_\mu(x)$ represents a potential well whose depth is controlled by μ and $V_\mu(x)$ is called a steep potential well if μ is large. It is worth mentioning that we do not impose any other hypotheses on the behavior of $g(x)$ for $|x|\to\infty$.

The main results in this paper show that the following Kirchhoff-type equation

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u+u=\left(\frac{1}{|x|^\alpha}*|u|^p\right)|u|^{p-2}u,\\ u\in H_0^1(\Omega)\end{cases} \tag{$\mathcal{P}_\infty$}$$

plays a special role: it can be seen as the limit problem for $(\mathcal{P}_\mu)$ as $\mu\to+\infty$, where $\Omega=\operatorname{int} g^{-1}(0)$.