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Monge problem in metric measure spaces with Riemannian curvature-dimension condition

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ABSTRACT

We prove the existence of solutions for the Monge minimization problem, addressed in a metric measure space (X, d, m) enjoying the Riemannian curvature-dimension condition $\mathsf{RCD}^*(K, N)$, with $N < \infty$. For the first marginal measure, we assume that $\mu_0 \ll m$. As a corollary, we obtain that the Monge problem and its relaxed version, the Monge–Kantorovich problem, attain the same minimal value.

Moreover we prove a structure theorem for d-cyclically monotone sets: neglecting a set of zero m-measure they do not contain any branching structures, that is, they can be written as the disjoint union of the image of a disjoint family of geodesics.

1. Introduction

Let (X, d, m) be a metric measure space verifying the Riemannian curvature dimension condition $\mathsf{RCD}^*(K, N)$ for $K, N \in \mathbb{R}$ with $N \geq 1$. In this note we prove the existence of a solution for the following Monge problem: given $\mu_0, \mu_1 \in \mathcal{P}(X)$, the space of Borel probability measures over X, solve the following minimization problem

$$\inf_{T_\sharp \mu_0 = \mu_1} \int_X d(x, T(x)) \mu_0(dx), \tag{1.1}$$

provided $\mu_0 \ll m$. In more detail, the minimization of the functional runs over the set of μ_0-measurable maps $T : X \to X$ such that $T_\sharp \mu_0 = \mu_1$, that is

$$\mu_0(T^{-1}(A)) = \mu_1(A), \quad \forall A \in \mathcal{B}(X),$$

where $\mathcal{B}(X)$ denotes the σ-algebra of all Borel subsets of X.

On the way to the proof of the existence of an optimal map, we will also prove a structure theorem for branching structures inside d-cyclically monotone sets. Before giving the statements of the two main results of this note and an account on the strategies to prove them, we recall some of the (extensive) literature on the Monge minimization problem.

The first formulation for (1.1) (Monge in 1781) was addressed in $\mathbb{R}^n$ with the cost given by the Euclidean norm and the measures $\mu_0, \mu_1 \ll \mathcal{L}^n$ were supposed to be supported on two disjoint compact sets. The original problem remained unsolved for a long time. In 1978 Sudakov in [1] proposed a solution for any distance cost induced by a norm, but an argument about disintegration of measures contained in his proof was not correct, see [2] for details. Then the Euclidean case was correctly solved by Evans and Gangbo in [3], under the assumptions that $\operatorname{spt} \mu_0 \cap \operatorname{spt} \mu_1 = \emptyset$, $\mu_0, \mu_1 \ll \mathcal{L}^n$ and their densities are Lipschitz functions with compact support. After that, many results reduced the assumptions on the supports of μ_0, μ_1, see [4,5]. The result on manifolds with geodesic cost is obtained in [6]. The case of a general norm as cost function

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on $\mathbb{R}^n$ has been solved first in the particular case of crystalline norms in [7], and then in full generality independently by L. Caravenna in [8] and by T. Champion and L. De Pascale in [9].

The study of the geodesic metric space framework started with [10], where the metric space was assumed to be also non-branching. There the existence of solutions to (1.1) was obtained for metric spaces verifying the measure-contraction property $\mathsf{MCP}(K, N)$ (for instance the Heisenberg group). An application of the results of [10] to the Wiener space can be found in [11]. Then in [12] the problem was studied removing the non-branching assumption but obtaining existence of solutions only in a particular case.

Non-branching metric measure spaces enjoying $\mathsf{CD}^*(K, N)$ also verify $\mathsf{MCP}(K, N)$, see [13]. Then from [10] the Monge problem is solved also in that case. So with respect to the most general known case, we impose a stronger curvature information (namely $\mathsf{RCD}^*(K, N)$) and we remove the non-branching assumption.

1.1. The results

The nowadays classical strategy to show existence of optimal maps is to relax the integral functional to the larger class of transport plans

$$\Pi(\mu_0, \mu_1) := \{\pi \in \mathcal{P}(X \times X) : (P_1)_\sharp \pi = \mu_0, (P_2)_\sharp \pi = \mu_1\},$$

over where the functional we want to minimize has now the following expression

$$\int d(x, y)\eta(dxdy).$$

For $i = 1, 2$, $P_i : X \times X \to X$ denotes the projection map on the ith component. Assuming that the functional is finite at least on one element of $\Pi(\mu_0, \mu_1)$, we have the existence of $\eta_{opt} \in \Pi(\mu_0, \mu_1)$ so that

$$\int d(x, y)\eta_{opt}(dxdy) = \inf_{\eta \in \Pi(\mu_0,\mu_1)} \int_X d(x, y)\eta(dx),$$

by linearity in η and tightness of $\Pi(\mu_0, \mu_1)$. Then the central question, whose positive answer would prove existence of a solution to Monge problem, is whether η_{opt} is supported on the graph of a m-measurable map $T : X \to X$.

A property of η_{opt} inside $\Pi(\mu_0, \mu_1)$ is the fact that is concentrated on a d-cyclically monotone set. We shall build an optimal map starting from this monotonicity. But while the Riemannian curvature-dimension condition $\mathsf{RCD}^*(K, N)$ gives crucial information on d^2-cyclically monotone sets (neglecting a set of measure zero, they are the graph of a measurable map, see Section 2 and references therein), nothing is known under this curvature assumption on the structure of d-cyclically monotone sets. In particular what we would like to exclude is the presence of branching structures. Note that the first result proving absence of branching geodesics assuming a curvature condition, in that case strong $\mathsf{CD}(K, \infty)$, is contained in [14]. The same type of result, but only for L^2-Wasserstein geodesics with end point a Dirac delta, was already present in an earlier work of Rajala, see [15].

The strategy we will follow is: prove that d-cyclically monotone sets do not have branching structures m-almost everywhere; then use the approach with Disintegration Theorem (see for instance [10] and references therein) to reduce the Monge problem to a family of 1-dimensional Monge problem. There one can apply the 1-dimensional theory. Thanks to the curvature assumption we can prove a suitable property for the first marginal measures and obtain the existence of the 1-dimensional optimal maps, one for each 1-dimensional Monge problem. Then gluing together all the one-dimensional optimal maps, one gets an optimal map $T : X \to X$ solving the Monge problem (1.1). A more precise program on the use of Disintegration Theorem in the Monge problem will be given in Section 3.

We conclude this introductory part stating the two main results we will prove. The first is about the structure of the d-cyclically monotone set associated to a Kantorovich potential φ^d for the problem (1.1).

Theorem 1.1. *Let (X, d, m) be a metric measure space verifying $\mathsf{RCD}^*(K, N)$ for some $K, N \in \mathbb{R}$, with $N \geq 1$. Let moreover Γ be a d-cyclically monotone set as (3.1) and let $\mathcal{T}_e$ be the set of all points moved by Γ as in Definition 3.2. Then there exists $\mathcal{T} \subset \mathcal{T}_e$ that we call the transport set such that*

$$m(\mathcal{T}_e \setminus \mathcal{T}) = 0,$$

and for all $x \in \mathcal{T}$, the transport ray $R(x)$ is formed by a single geodesic and for $x \neq y$, both in $\mathcal{T}$, either $R(x) = R(y)$ or $R(x) \cap R(y)$ is contained in the set of initial points $a \cup b$ as defined in Definition 3.2.

All the terminology used in Theorem 1.1 will be introduced in Section 3. Taking advantage of Theorem 1.1 we then obtain the following.

Theorem 1.2. *Let (X, d, m) be a metric measure space verifying $\mathsf{RCD}^*(K, N)$ for $N < \infty$. Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ with $W_1(\mu_0, \mu_1) < \infty$ and $\mu_0 \ll m$. Then there exists a Borel map $T : X \to X$ such that $T_\sharp \mu_0 = \mu_1$ and*

$$\int_X d(x, T(x))\mu_0(dx) = \int_{X\times X} d(x, y)\eta_{opt}(dxdy).$$

In the previous theorem, W_1 denotes the L^1-Wasserstein distance on the space of probability measures on (X, d).